

# Parabolic integrodifferential identification problems related to radial memory kernels I\*

Alberto Favaron (Milan), Alfredo Lorenzi (Milan)<sup>†</sup>

**Abstract.** We are concerned with the problem of recovering the radial kernel  $k$ , depending also on time, in the parabolic integro-differential equation

$$D_t u(t, x) = \mathcal{A}u(t, x) + \int_0^t k(t-s, |x|) \mathcal{B}u(s, x) ds + \int_0^t D_{|x|} k(t-s, |x|) \mathcal{C}u(s, x) ds + f(t, x),$$

$\mathcal{A}$  being a uniformly elliptic second-order linear operator in divergence form. We single out a special class of operators  $\mathcal{A}$  and two pieces of suitable additional information for which the problem of identifying  $k$  can be uniquely solved locally in time when the domain under consideration is a spherical corona or an annulus.

*2000 Mathematical Subject Classification.* Primary 45Q05. Secondary 45K05, 45N05, 35K20, 35K90.

*Key words and phrases.* Identification problems. Parabolic integro-differential equations in two and three space dimensions. Recovering radial kernels depending also on time. Existence and uniqueness results.

## 1 Posing the identification problem

The present paper is strictly related to the previous work [CL] by the latter author and F. Colombo. Indeed, the problem we are going to investigate consists in identifying an unknown radial memory kernel  $k$  also depending on time, which appears in the following integro-differential equation related to the spherical corona  $\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : R_1 < |x| < R_2\}$ , where  $0 < R_1 < R_2$  and  $|x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ :

$$D_t u(t, x) = \mathcal{A}u(t, x) + \int_0^t k(t-s, |x|) \mathcal{B}u(s, x) ds + \int_0^t D_{|x|} k(t-s, |x|) \mathcal{C}u(s, x) ds + f(t, x),$$

$$\forall (t, x) \in [0, T] \times \Omega. \quad (1.1)$$

In equation (1.1)  $\mathcal{A}$  and  $\mathcal{B}$  are two second-order linear differential operators, while  $\mathcal{C}$  is a first-order differential operator, having respectively the following forms:

$$\mathcal{A} = \sum_{j=1}^3 D_{x_j} \left( \sum_{k=1}^3 a_{j,k}(x) D_{x_k} \right), \quad \mathcal{B} = \sum_{j=1}^3 D_{x_j} \left( \sum_{k=1}^3 b_{j,k}(x) D_{x_k} \right), \quad \mathcal{C} = \sum_{j=1}^3 c_j(x) D_{x_j}. \quad (1.2)$$

\*Work partially supported by the Italian Ministero dell'Università e della Ricerca Scientifica e Tecnologica (M.U.R.S.T.).

<sup>†</sup>The authors are members of G.N.A.M.P.A. of the Italian Istituto Nazionale di Alta Matematica (I.N.d.A.M.).

In addition operator  $\mathcal{A}$  is uniformly elliptic, i.e. there exist two positive constants  $\alpha_1$  and  $\alpha_2$  with  $\alpha_1 \leq \alpha_2$  such that

$$\alpha_1 |\xi|^2 \leq \sum_{j,k=1}^3 a_{j,k}(x) \xi_j \xi_k \leq \alpha_2 |\xi|^2, \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^3. \quad (1.3)$$

Before going on, we note that, to the authors' knowledge, the recover of a kernel  $k$  depending also on spatial variables is a quite new problem, as far as first-order in time integro-differential equations are concerned. We can quote, besides [CL], the papers [JW] and [JJ] that are one-dimensional in character, since not only the kernel  $k$  is assumed to be *degenerate*, i.e. of the form  $k(t, x) = \sum_{j=1}^N m_j(t) \mu_j(x)$ , but also the space-dependent functions  $\mu_j$ ,  $j = 1, \dots, N$ , are assumed to be *known*. As a consequence, the identification problem reduces to recovering the  $N$  unknown time-dependent functions  $m_j$ ,  $j = 1, \dots, N$ . This latter is nowadays a classical (vector-) identification problem.

Coming back to our problem, since the domain  $\Omega$  has a radial symmetry, we will use the classical spherical co-ordinates  $(r, \varphi, \theta) \in (0, \infty) \times (0, 2\pi) \times (0, \pi)$  related to the Cartesian ones by the well-known relationship:

$$(x_1, x_2, x_3) = (r \cos \varphi \sin \theta, r \sin \varphi \sin \theta, r \cos \theta). \quad (1.4)$$

Then we prescribe the *initial condition*

$$u(0, x) = u_0(x), \quad \forall x \in \Omega, \quad (1.5)$$

where  $u_0 : \overline{\Omega} \rightarrow \mathbb{R}$  is a given smooth function, as well as one of the following boundary value conditions, where  $u_1 : [0, T] \times \overline{\Omega} \rightarrow \mathbb{R}$  is a prescribed smooth function, too;

$$(D,D) \quad u(t, x) = u_1(t, x), \quad \forall (t, x) \in [0, T] \times \partial\Omega, \quad (1.6)$$

$$(D,N) \quad \begin{cases} u(t, x) = u_1(t, x), & \forall (t, x) \in [0, T] \times \partial B(0, R_2), \\ \frac{\partial u}{\partial \nu}(t, x) = \frac{\partial u_1}{\partial \nu}(t, x), & \forall (t, x) \in [0, T] \times \partial B(0, R_1), \end{cases} \quad (1.7)$$

$$(N,D) \quad \begin{cases} \frac{\partial u}{\partial \nu}(t, x) = \frac{\partial u_1}{\partial \nu}(t, x), & \forall (t, x) \in [0, T] \times \partial B(0, R_2), \\ u(t, x) = u_1(t, x), & \forall (t, x) \in [0, T] \times \partial B(0, R_1), \end{cases} \quad (1.8)$$

$$(N,N) \quad \frac{\partial u}{\partial \nu}(t, x) = \frac{\partial u_1}{\partial \nu}(t, x), \quad \forall (t, x) \in [0, T] \times \partial\Omega. \quad (1.9)$$

Here D and N stand, respectively, for the Dirichlet and the conormal boundary conditions, where the conormal vector  $\nu$  is defined by  $\nu(x) = \sum_{j,k=1}^3 a_{j,k}(x) n_k(x)$ ,  $n(x)$  denoting the outwarding normal vector at  $x \in \partial\Omega$ .

To determine the radial memory kernel  $k$  we need also the two following additional pieces of information:

$$\Phi[u(t, \cdot)](r) := g_1(t, r), \quad \forall (t, r) \in [0, T] \times (R_1, R_2), \quad (1.10)$$

$$\Psi[u(t, \cdot)] := g_2(t), \quad \forall t \in [0, T], \quad (1.11)$$

where  $\Phi$  is a linear operator acting on the angular variables  $\varphi, \theta$  only, while  $\Psi$  is a linear operator acting on all the space variables  $r, \varphi, \theta$ .

*Convention:* from now on we will denote by  $P(H,K)$ ,  $H,K \in \{D,N\}$ , the identification problem consisting of (1.1), (1.5), the boundary condition (H,K) and (1.10), (1.11).

An example of admissible linear operators  $\Phi$  and  $\Psi$  is the following:

$$\Phi[v](r) := \int_0^\pi \sin\theta d\theta \int_0^{2\pi} \lambda(R_2 x') v(r x') d\varphi, \quad (1.12)$$

$$\Psi[v] := \int_{R_1}^{R_2} r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} \psi(r x') v(r x') d\varphi, \quad (1.13)$$

where  $x' = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$ , while  $\lambda : \partial B(0, R_2) \rightarrow \mathbb{R}$  and  $\psi : \overline{\Omega} \rightarrow \mathbb{R}$  are two smooth assigned functions.

From (1.6)–(1.11) we (formally) deduce that our data have to satisfy the following consistency conditions, respectively:

$$(C1,D,D) \quad u_0(x) = u_1(0, x), \quad \forall x \in \partial\Omega, \quad (1.14)$$

$$(C1,D,N) \quad \begin{cases} u_0(x) = u_1(0, x), & \forall x \in \partial B(0, R_2), \\ \frac{\partial u_0}{\partial \nu}(x) = \frac{\partial u_1}{\partial \nu}(0, x), & \forall x \in \partial B(0, R_1), \end{cases} \quad (1.15)$$

$$(C1,N,D) \quad \begin{cases} \frac{\partial u_0}{\partial \nu}(x) = \frac{\partial u_1}{\partial \nu}(0, x), & \forall x \in \partial B(0, R_2), \\ u_0(x) = u_1(0, x), & \forall x \in \partial B(0, R_1), \end{cases} \quad (1.16)$$

$$(C1,N,N) \quad \frac{\partial u_0}{\partial \nu}(x) = \frac{\partial u_1}{\partial \nu}(0, x), \quad \forall x \in \partial\Omega, \quad (1.17)$$

$$\Phi[u_0](r) = g_1(0, r), \quad \forall r \in (R_1, R_2), \quad (1.18)$$

$$\Psi[u_0] = g_2(0). \quad (1.19)$$

Though our identification problem seems to be a very simple generalization of that in the quoted paper [CL] to the case where the kernel and the domain  $\Omega$  are assumed to have radial symmetries, it should be noted that the choice of  $\Omega$  coinciding with a ball, which seems to be the most natural, gives rise to a lot of technical difficulties. As a consequence, such a problem, in its generality, is still open. Here we stress only that the mathematical difficulties are concentrated at the centre of the ball (cf. Remark 2.9).

However, also in the case of a spherical corona, to solve our identification problem, we are forced to restrict the *admissible* operators  $\mathcal{A}$  to those satisfying, in addition to (1.2), (1.3) also the following condition for some function  $h \in C([R_1, R_2])$ :

$$\sum_{j,k=1}^3 x_j x_k a_{j,k}(x) = |x|^2 h(|x|), \quad \forall x \in \overline{\Omega}. \quad (1.20)$$

We conclude this section by a remark on radial solutions to our identification problem.

**Remark 1.1.** Assume that  $\mathcal{A} = \mathcal{B} = \Delta_n$ ,  $n = 2, 3$ ,  $\mathcal{C} = D_{|x|}$  and  $f$  and  $u_0$  are radial functions. We shall refer to this as to the “radial case”. If we assumed, as at first glance seems reasonable, that in our identification problem also the state function  $u$ , i.e. the temperature in physical applications, should be radial, then definition (1.12) would reduce to the form

$$\Phi[v](r) := C_0 v(r), \quad \forall r \in [R_1, R_2], \quad (1.21)$$

$C_0$  being a non-zero constant. As a consequence, the additional condition (1.10) would amount to requiring that  $u$  itself should be *a priori* known. If this is not the case and we need to determine both  $u$  and  $k$ , we are led to assume that either of the functions  $f$  or  $u_0$  is *not radial*.

On the contrary, if in the “radial case” we *a priori* knew a radial state function  $u$ , then our problem would reduce to the following Volterra integro-differential equation of the first kind, where  $n = 2, 3$ :

$$\int_0^t \{D_r k(t-s, r) D_r u(s, r) + k(t-s, r) [D_r^2 u(s, r) + (n-1)r^{-1} D_r u(s, r)]\} ds = \tilde{f}(t, r),$$

$$\forall (t, r) \in [0, T] \times [R_1, R_2]. \quad (1.22)$$

Of course the right hand-side  $\tilde{f}$  must satisfy the consistency condition

$$\tilde{f}(0, r) = 0, \quad \forall r \in [R_1, R_2].$$

Furthermore, we note that in this case condition (1.11) with  $\Psi$  being defined by (1.13) makes no sense, since  $\Psi$  reduces to  $C_1 \int_{R_1}^{R_2} \tilde{\psi}(r) u(t, r) dr$ ,  $C_1$  being a constant, i.e. to a *known fixed function independent of  $k$* !

However, by differentiation with respect to time of both sides, equation (1.22) turns into the equivalent one

$$\int_0^t \{D_r k(t-s, r) D_t D_r u(s, r) + k(t-s, r) [D_t D_r^2 u(s, r) + (n-1)r^{-1} D_t D_r u(s, r)]\} ds$$

$$+ D_r k(t, r) D_r u(0, r) + k(t, r) [D_r^2 u(0, r) + (n-1)r^{-1} D_r u(0, r)] = D_t \tilde{f}(t, r),$$

$$\forall (t, r) \in [0, T] \times [R_1, R_2]. \quad (1.23)$$

To solve this equation we need, e.g., an additional information of the form

$$\int_{R_1}^{R_2} \lambda(r) k(t, r) dr = h(t), \quad \forall t \in [0, T]. \quad (1.24)$$

Using the same decomposition for  $k$  as in section 3, in section 7 we will solve the less usual system (1.23) and (1.24).

## 2 Main results

In this section we state our *local in time* existence and uniqueness result related to the identification problem P(H,K). For this purpose we assume that the coefficients of operators  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$  satisfy, in addition to (1.3) also the following properties:

$$a_{i,j} \in W^{2,\infty}(\Omega), \quad a_{i,j} = a_{j,i}, \quad i, j = 1, 2, 3, \quad (2.1)$$

$$b_{i,j} \in W^{1,\infty}(\Omega), \quad c_i \in L^\infty(\Omega), \quad i, j = 1, 2, 3. \quad (2.2)$$

Hence, owing to (1.20) we get that the function

$$\begin{aligned} & [\tilde{a}_{1,1}(r, \varphi, \theta) \cos^2 \varphi + \tilde{a}_{1,2}(r, \varphi, \theta) \sin 2\varphi + \tilde{a}_{2,2}(r, \varphi, \theta) \sin^2 \varphi] \sin^2 \theta \\ & + [\tilde{a}_{1,3}(r, \varphi, \theta) \cos \varphi + \tilde{a}_{2,3}(r, \varphi, \theta) \sin \varphi] \sin 2\theta + \tilde{a}_{3,3}(r, \varphi, \theta) \cos^2 \theta := h(r) \end{aligned} \quad (2.3)$$

depends only on the variable  $r$  where we have set

$$\tilde{a}_{i,j}(r, \varphi, \theta) = a_{i,j}(r \cos \varphi \sin \theta, r \sin \varphi \sin \theta, r \cos \theta).$$

**Remark 2.1.** To show that our previous condition (2.3) is meaningful we exhibit a class of coefficients  $a_{i,j}$  satisfying such a property. Let us suppose that there exist three functions  $a, b, d \in W^{2,\infty}(R_1, R_2)$  and a function  $c \in W^{2,\infty}(\Omega)$ ,  $a$  and  $c$  being, respectively, *positive* and *non-negative*, such that

$$\left\{ \begin{aligned} a_{1,1}(x) &= a(|x|) + \frac{(x_2^2 + x_3^2)[c(x) - b(|x|)]}{|x|^2} + \frac{x_1^2 d(|x|)}{|x|^2}, \\ a_{2,2}(x) &= a(|x|) + \frac{(x_1^2 + x_3^2)[c(x) - b(|x|)]}{|x|^2} + \frac{x_2^2 d(|x|)}{|x|^2}, \\ a_{3,3}(x) &= a(|x|) + \frac{(x_1^2 + x_2^2)[c(x) - b(|x|)]}{|x|^2} + \frac{x_3^2 d(|x|)}{|x|^2}, \\ a_{j,k}(x) &= a_{k,j}(x) = \frac{x_j x_k [b(|x|) - c(x) + d(|x|)]}{|x|^2}, \quad 1 \leq j, k \leq 3, \quad j \neq k. \end{aligned} \right. \quad (2.4)$$

Simple computations show that property (1.20) is satisfied with  $h(r) = a(r) + d(r)$ .

Denoting by  $f^+$ ,  $f^-$ , respectively, the positive and the negative parts of a function  $f$ , for every  $x \in \Omega$  and  $\xi \in \mathbb{R}^3$  from (2.4) it follows

$$\begin{aligned} \sum_{j,k=1}^3 a_{j,k}(x) \xi_j \xi_k &= a(|x|) |\xi|^2 + \frac{c(x) - b(|x|)}{|x|^2} [(x_2^2 + x_3^2) \xi_1^2 + (x_1^2 + x_3^2) \xi_2^2 + (x_1^2 + x_2^2) \xi_3^2 \\ &\quad - 2x_1 x_2 \xi_1 \xi_2 - 2x_1 x_3 \xi_1 \xi_3 - 2x_2 x_3 \xi_2 \xi_3] + \frac{d(|x|)}{|x|^2} (x_1 \xi_1 + x_2 \xi_2 + x_3 \xi_3)^2 \\ &\geq a(|x|) |\xi|^2 - \frac{b^+(|x|)}{|x|^2} |x \wedge \xi|^2 - \frac{d^-(|x|)}{|x|^2} [x \cdot \xi]^2 \geq [a(|x|) - b^+(|x|) - d^-(|x|)] |\xi|^2, \end{aligned} \quad (2.5)$$

where  $\wedge$  and  $\cdot$  denote, respectively, the wedge and the inner product in  $\mathbb{R}^3$ .

Hence, to ensure the uniform ellipticity of operator  $\mathcal{A}$ , we need the additional assumption:

$$a(r) - b^+(r) - d^-(r) > 0, \quad \forall r \in [R_1, R_2]. \quad (2.6)$$

Condition (2.6) amounts to requiring that function  $a$  is large enough with respect to  $b$  and  $d^-$ . Consequently (1.3) is trivially satisfied owing to (2.6) with

$$\alpha_1 = \min_{r \in [R_1, R_2]} [a(r) - b^+(r) - d^-(r)], \quad \alpha_2 = \|a + b^- + d^+\|_{C([R_1, R_2])} + \|c\|_{C(\overline{\Omega})}.$$

**Remark 2.2.** We can widen the class of special operators in the previous remark introducing the following function sequence  $\{a_{i,j}^{(n)}\}_{n=1}^{+\infty}$ :

$$a_{1,1}^{(n)}(x) = 2x_1^{2n-2}x_2^{2n}x_3^{2n}, \quad a_{2,2}^{(n)}(x) = 2x_1^{2n}x_2^{2n-2}x_3^{2n}, \quad a_{3,3}^{(n)}(x) = 2x_1^{2n}x_2^{2n}x_3^{2n-2}, \quad (2.7)$$

$$a_{1,2}^{(n)}(x) = a_{2,1}^{(n)}(x) = -2x_1^{2n-1}x_2^{2n-1}x_3^{2n}, \quad (2.8)$$

$$a_{1,3}^{(n)}(x) = a_{3,1}^{(n)}(x) = -2x_1^{2n-1}x_2^{2n}x_3^{2n-1}, \quad (2.9)$$

$$a_{2,3}^{(n)}(x) = a_{3,2}^{(n)}(x) = -2x_1^{2n}x_2^{2n-1}x_3^{2n-1}. \quad (2.10)$$

Simple computations show that

$$\begin{aligned} \sum_{j,k=1}^3 a_{j,k}^{(n)}(x) \xi_j \xi_k &= (x_1 x_2 x_3)^{2n-2} [(\xi_1 x_2 x_3 - x_1 \xi_2 x_3)^2 + (x_1 \xi_2 x_3 - x_1 x_2 \xi_3)^2 \\ &\quad + (x_1 x_2 \xi_3 - \xi_1 x_2 x_3)^2] \geq 0, \quad \forall (x, \xi) \in \overline{\Omega} \times \mathbb{R}^3, \quad \forall n \in \mathbb{N}. \end{aligned} \quad (2.11)$$

In particular we get

$$\sum_{j,k=1}^3 x_j x_k a_{j,k}^{(n)}(x) = 0, \quad \forall x \in \overline{\Omega}, \quad \forall n \in \mathbb{N}. \quad (2.12)$$

Then with any sequence of *non-negative* functions  $\{d_n\}_{n=1}^{+\infty} \subset W^{2,\infty}(\Omega)$  we associate the special coefficients:

$$a_{j,k}(x) = \bar{a}_{j,k}(x) + \sum_{n=1}^N a_{j,k}^{(n)}(x) d_n(x), \quad x \in \overline{\Omega}, \quad j, k = 1, 2, 3, \quad N \in \mathbb{N}, \quad (2.13)$$

where the coefficients  $\bar{a}_{j,k}$  are defined by equations (2.4). Also the value  $N = +\infty$  is allowed provided that the series  $\sum_{n=1}^{+\infty} a_{j,k}^{(n)}(x) d_n(x)$  may be differentiated twice term by term, with a sum in  $W^{2,\infty}(\Omega)$ .

It is immediate to check that both property (1.20) and the uniform ellipticity are satisfied under the same assumptions as in the previous remark.

We have thus showed that the class of admissible coefficients is not limited to those represented by (2.4).

**Remark 2.3.** Simple computations show that, when the coefficients  $a_{j,k}$  are given by (2.4), then the conormal vector  $\nu^{(l)}$  on  $\partial B(0, R_l)$  coincides with the normal vector

$$\nu^{(l)}(x) = (-1)^l R_l^{-1} [a(R_l) - d(R_l)] x, \quad l = 1, 2.$$

In order to find out the right hypotheses on the linear operators  $\Phi$  and  $\Psi$ , it will be convenient to rewrite the operator  $\mathcal{A}$  in the spherical co-ordinates  $(r, \varphi, \theta)$ . Recall first that the gradient  $\nabla = (D_{x_1}, D_{x_2}, D_{x_3})$  can be expressed in terms of  $(D_r, D_\varphi, D_\theta)$  by the formulae:

$$\begin{cases} D_{x_1} = \cos \varphi \sin \theta D_r - \frac{\sin \varphi}{r \sin \theta} D_\varphi + \frac{\cos \varphi \cos \theta}{r} D_\theta, \\ D_{x_2} = \sin \varphi \sin \theta D_r + \frac{\cos \varphi}{r \sin \theta} D_\varphi + \frac{\sin \varphi \cos \theta}{r} D_\theta, \\ D_{x_3} = \cos \theta D_r - \frac{\sin \theta}{r} D_\theta. \end{cases} \quad (2.14)$$

As a consequence, simple computations easily yield

$$\sum_{k=1}^3 a_{1,k}(x)D_{x_k} = f_1(r, \varphi, \theta)D_r + \frac{f_2(r, \varphi, \theta)}{r \sin \theta}D_\varphi + \frac{f_3(r, \varphi, \theta)}{r}D_\theta, \quad (2.15)$$

$$\sum_{k=1}^3 a_{2,k}(x)D_{x_k} = g_1(r, \varphi, \theta)D_r + \frac{g_2(r, \varphi, \theta)}{r \sin \theta}D_\varphi + \frac{g_3(r, \varphi, \theta)}{r}D_\theta, \quad (2.16)$$

$$\sum_{k=1}^3 a_{3,k}(x)D_{x_k} = h_1(r, \varphi, \theta)D_r + \frac{h_2(r, \varphi, \theta)}{r \sin \theta}D_\varphi + \frac{h_3(r, \varphi, \theta)}{r}D_\theta, \quad (2.17)$$

functions  $f_j, g_j, h_j, j=1, 2, 3$ , being defined by

$$\begin{cases} f_1(r, \varphi, \theta) := \tilde{a}_{1,1}(r, \varphi, \theta)\cos\varphi \sin\theta + \tilde{a}_{1,2}(r, \varphi, \theta)\sin\varphi \sin\theta + \tilde{a}_{1,3}(r, \varphi, \theta)\cos\theta, \\ f_2(r, \varphi, \theta) := \tilde{a}_{1,2}(r, \varphi, \theta)\cos\varphi - \tilde{a}_{1,1}(r, \varphi, \theta)\sin\varphi, \\ f_3(r, \varphi, \theta) := \tilde{a}_{1,1}(r, \varphi, \theta)\cos\varphi \cos\theta + \tilde{a}_{1,2}(r, \varphi, \theta)\sin\varphi \cos\theta - \tilde{a}_{1,3}(r, \varphi, \theta)\sin\theta, \end{cases} \quad (2.18)$$

$$\begin{cases} g_1(r, \varphi, \theta) := \tilde{a}_{2,1}(r, \varphi, \theta)\cos\varphi \sin\theta + \tilde{a}_{2,2}(r, \varphi, \theta)\sin\varphi \sin\theta + \tilde{a}_{2,3}(r, \varphi, \theta)\cos\theta, \\ g_2(r, \varphi, \theta) := \tilde{a}_{2,2}(r, \varphi, \theta)\cos\varphi - \tilde{a}_{2,1}(r, \varphi, \theta)\sin\varphi, \\ g_3(r, \varphi, \theta) := \tilde{a}_{2,1}(r, \varphi, \theta)\cos\varphi \cos\theta + \tilde{a}_{2,2}(r, \varphi, \theta)\sin\varphi \cos\theta - \tilde{a}_{2,3}(r, \varphi, \theta)\sin\theta, \end{cases} \quad (2.19)$$

$$\begin{cases} h_1(r, \varphi, \theta) := \tilde{a}_{3,1}(r, \varphi, \theta)\cos\varphi \sin\theta + \tilde{a}_{3,2}(r, \varphi, \theta)\sin\varphi \sin\theta + \tilde{a}_{3,3}(r, \varphi, \theta)\cos\theta, \\ h_2(r, \varphi, \theta) := \tilde{a}_{3,2}(r, \varphi, \theta)\cos\varphi - \tilde{a}_{3,1}(r, \varphi, \theta)\sin\varphi, \\ h_3(r, \varphi, \theta) := \tilde{a}_{3,1}(r, \varphi, \theta)\cos\varphi \cos\theta + \tilde{a}_{3,2}(r, \varphi, \theta)\sin\varphi \cos\theta - \tilde{a}_{3,3}(r, \varphi, \theta)\sin\theta. \end{cases} \quad (2.20)$$

Hence, using again (2.14) and applying it to relations (2.15)–(2.17), we get:

$$\begin{aligned} D_{x_1} \left( \sum_{k=1}^3 a_{1,k}(x)D_{x_k} \right) &= D_r \left[ f_1(r, \varphi, \theta)\cos\varphi \sin\theta D_r + \frac{f_2(r, \varphi, \theta)\cos\varphi}{r}D_\varphi \right. \\ &\quad \left. + \frac{f_3(r, \varphi, \theta)\cos\varphi \sin\theta}{r}D_\theta \right] - \frac{\sin\varphi}{r \sin\theta}D_\varphi \left[ f_1(r, \varphi, \theta)D_r + \frac{f_2(r, \varphi, \theta)}{r \sin\theta}D_\varphi + \frac{f_3(r, \varphi, \theta)}{r}D_\theta \right] \\ &\quad + \frac{\cos\theta}{r}D_\theta \left[ f_1(r, \varphi, \theta)\cos\varphi D_r + \frac{f_2(r, \varphi, \theta)\cos\varphi}{r \sin\theta}D_\varphi + \frac{f_3(r, \varphi, \theta)\cos\varphi}{r}D_\theta \right], \end{aligned} \quad (2.21)$$

$$\begin{aligned} D_{x_2} \left( \sum_{k=1}^3 a_{2,k}(x)D_{x_k} \right) &= D_r \left[ g_1(r, \varphi, \theta)\sin\varphi \sin\theta D_r + \frac{g_2(r, \varphi, \theta)\sin\varphi}{r}D_\varphi \right. \\ &\quad \left. + \frac{g_3(r, \varphi, \theta)\sin\varphi \sin\theta}{r}D_\theta \right] + \frac{\cos\varphi}{r \sin\theta}D_\varphi \left[ g_1(r, \varphi, \theta)D_r + \frac{g_2(r, \varphi, \theta)}{r \sin\theta}D_\varphi + \frac{g_3(r, \varphi, \theta)}{r}D_\theta \right] \\ &\quad + \frac{\cos\theta}{r}D_\theta \left[ g_1(r, \varphi, \theta)\sin\varphi D_r + \frac{g_2(r, \varphi, \theta)\sin\varphi}{r \sin\theta}D_\varphi + \frac{g_3(r, \varphi, \theta)\sin\varphi}{r}D_\theta \right], \end{aligned} \quad (2.22)$$

$$D_{x_3} \left( \sum_{k=1}^3 a_{3,k} D_{x_k} \right) = D_r \left[ h_1(r, \varphi, \theta) \cos \theta D_r + \frac{h_2(r, \varphi, \theta) \cos \theta}{r \sin \theta} D_\varphi + \frac{h_3(r, \varphi, \theta) \cos \theta}{r} D_\theta \right] \\ - \frac{\sin \theta}{r} D_\theta \left[ \frac{h_1(r, \varphi, \theta)}{r} D_r + \frac{h_2(r, \varphi, \theta)}{r \sin \theta} D_\varphi + \frac{h_3(r, \varphi, \theta)}{r} D_\theta \right]. \quad (2.23)$$

Let us now define the following functions, where  $j = 1, 2, 3$ :

$$\begin{cases} k_j(r, \varphi, \theta) := f_j(r, \varphi, \theta) \cos \varphi \sin \theta + g_j(r, \varphi, \theta) \sin \varphi \sin \theta + h_j(r, \varphi, \theta) \cos \theta, \\ l_j(r, \varphi, \theta) := f_j(r, \varphi, \theta) \cos \varphi + g_j(r, \varphi, \theta) \sin \varphi. \end{cases} \quad (2.24)$$

Using (2.18)–(2.20) one can easily check that  $k_1$  coincides with the function  $h$  defined in (2.3). Therefore, by virtue of the assumptions (1.20) made on the coefficients  $a_{i,j}$ , we conclude that  $k_1$  depends on  $r$  only.

Then, rearranging the terms on the right-hand sides of (2.21)–(2.23), we obtain the following polar representation  $\tilde{\mathcal{A}}$  for the second-order differential operator  $\mathcal{A}$ :

$$\begin{aligned} \tilde{\mathcal{A}} = & D_r \left[ k_1(r) D_r + \frac{k_2(r, \varphi, \theta)}{r \sin \theta} D_\varphi + \frac{k_3(r, \varphi, \theta)}{r} D_\theta \right] - \frac{\sin \varphi}{r \sin \theta} D_\varphi \left[ f_1(r, \varphi, \theta) D_r \right. \\ & + \frac{f_2(r, \varphi, \theta)}{r \sin \theta} D_\varphi + \frac{f_3(r, \varphi, \theta)}{r} D_\theta \left. \right] + \frac{\cos \varphi}{r \sin \theta} D_\varphi \left[ g_1(r, \varphi, \theta) D_r + \frac{g_2(r, \varphi, \theta)}{r \sin \theta} D_\varphi \right. \\ & + \frac{g_3(r, \varphi, \theta)}{r} D_\theta \left. \right] - \frac{\sin \theta}{r} D_\theta \left[ h_1(r, \varphi, \theta) D_r + \frac{h_2(r, \varphi, \theta)}{r \sin \theta} D_\varphi + \frac{h_3(r, \varphi, \theta)}{r} D_\theta \right] \\ & + \frac{\cos \theta}{r} D_\theta \left[ l_1(r, \varphi, \theta) D_r + \frac{l_2(r, \varphi, \theta)}{r \sin \theta} D_\varphi + \frac{l_3(r, \varphi, \theta)}{r} D_\theta \right]. \end{aligned} \quad (2.25)$$

We can now list our requirements on operators  $\Phi$  and  $\Psi$  in accordance with the explicit case (1.12) and (1.13). We will work in Sobolev spaces related to  $L^p(\Omega)$  with

$$p \in (3, +\infty) \quad (2.26)$$

and we will assume

$$\Phi \in \mathcal{L}(L^p(\Omega), L^p(R_1, R_2)), \quad \Psi \in L^p(\Omega)^*, \quad (2.27)$$

$$\Phi[wu] = w \Phi[u], \quad \forall (w, u) \in L^p(R_1, R_2) \times L^p(\Omega), \quad (2.28)$$

$$D_r \Phi[u](r) = \Phi[D_r u](r), \quad \forall u \in W^{1,p}(\Omega) \text{ and } r \in (R_1, R_2), \quad (2.29)$$

$$\Phi \tilde{\mathcal{A}} = \tilde{\mathcal{A}}_1 \Phi + \Phi_1 \quad \text{on } W^{2,p}(\Omega), \quad \Phi_1 \in \mathcal{L}(W^{1,p}(\Omega), L^p(R_1, R_2)), \quad (2.30)$$

$$\Psi \tilde{\mathcal{A}} = \Psi_1 \quad \text{on } W^{2,p}(\Omega), \quad \Psi_1 \in W^{1,p}(\Omega)^*, \quad (2.31)$$

where

$$\tilde{\mathcal{A}}_1 = D_r[k_1(r) D_r]. \quad (2.32)$$

To state our result concerning the identification problem (1.1), (1.5)–(1.11) we need to

list also the following assumptions on the data  $f, u_0, u_1, g_1, g_2$ :

$$f \in C^{1+\beta}([0, T]; L^p(\Omega)), \quad f(0, \cdot) \in W^{2,p}(\Omega) \quad (2.33)$$

$$u_0 \in W^{4,p}(\Omega), \quad \mathcal{B}u_0 \in W_{\text{H,K}}^{2\delta,p}(\Omega) \quad (2.34)$$

$$u_1 \in C^{2+\beta}([0, T]; L^p(\Omega)) \cap C^{1+\beta}([0, T]; W^{2,p}(\Omega)), \quad (2.35)$$

$$\mathcal{A}u_0 + f(0, \cdot) - D_t u_1(0, \cdot) \in W_{\text{H,K}}^{2,p}(\Omega), \quad (2.36)$$

$$F := k'_0 \mathcal{C}u_0 + k_0 \mathcal{B}u_0 + \mathcal{A}^2 u_0 + \mathcal{A}f(0, \cdot) - D_t^2 u_1(0, \cdot) + D_t f(0, \cdot) \in W_{\text{H,K}}^{2\beta,p}(\Omega), \quad (2.37)$$

$$g_1 \in C^{2+\beta}([0, T]; L^p(R_1, R_2)) \cap C^{1+\beta}([0, T]; W^{2,p}(R_1, R_2)), \quad (2.38)$$

$$g_2 \in C^{2+\beta}([0, T]; \mathbb{R}), \quad (2.39)$$

where  $\beta \in (0, 1/2) \setminus \{1/(2p)\}$ ,  $\delta \in (\beta, 1/2) \setminus \{1/(2p)\}$ , and function  $k_0$  in (2.37) is defined by formula (3.18). Moreover, the spaces  $W_{\text{H,K}}^{2,p}(\Omega)$ ,  $\text{H,K} \in \{\text{D,N}\}$ , are defined by

$$W_{\text{H,K}}^{2,p}(\Omega) = \{w \in W^{2,p}(\Omega) : w \text{ satisfies the homogeneous condition (H,K)}\}, \quad (2.40)$$

whereas the spaces  $W_{\text{H,K}}^{2\gamma,p}(\Omega) \equiv (L^p(\Omega), W_{\text{H,K}}^{2,p}(\Omega))_{\gamma,\infty}$ ,  $\gamma \in (0, 1/2] \setminus \{1/(2p)\}$ , are interpolation spaces between  $W_{\text{H,K}}^{2,p}(\Omega)$  and  $L^p(\Omega)$  and they are defined (cf. [TR, section 4.3.3]), respectively, by the following equations:

$$W_{\text{D,D}}^{2\gamma,p}(\Omega) = \begin{cases} W^{2\gamma,p}(\Omega), & \text{if } 0 < \gamma < 1/(2p), \\ \{u \in W^{2\gamma,p}(\Omega) : u = 0 \text{ on } \partial\Omega\}, & \text{if } 1/(2p) < \gamma \leq 1/2, \end{cases} \quad (2.41)$$

$$W_{\text{D,N}}^{2\gamma,p}(\Omega) = \begin{cases} W^{2\gamma,p}(\Omega), & \text{if } 0 < \gamma < 1/(2p), \\ \{u \in W^{2\gamma,p}(\Omega) : u = 0 \text{ on } \partial B(0, R_2)\}, & \text{if } 1/(2p) < \gamma \leq 1/2, \end{cases} \quad (2.42)$$

$$W_{\text{N,D}}^{2\gamma,p}(\Omega) = \begin{cases} W^{2\gamma,p}(\Omega), & \text{if } 0 < \gamma < 1/(2p), \\ \{u \in W^{2\gamma,p}(\Omega) : u = 0 \text{ on } \partial B(0, R_1)\}, & \text{if } 1/(2p) < \gamma \leq 1/2, \end{cases} \quad (2.43)$$

$$W_{\text{N,N}}^{2\gamma,p}(\Omega) = W^{2\gamma,p}(\Omega), \quad \text{if } 0 < \gamma \leq 1/2. \quad (2.44)$$

**Remark 2.4.** Observe that our choice  $p \in (3, +\infty)$  implies the embeddings (cf. [AD, Theorem 5.4])

$$W^{1,p}(\Omega) \hookrightarrow C^{1-3/p}(\overline{\Omega}), \quad W_{\text{H,K}}^{2,p}(\Omega) \hookrightarrow C^{2-3/p}(\overline{\Omega}). \quad (2.45)$$

Assume also that  $u_0$  satisfies the following conditions for some positive constant  $m$ :

$$J_0(u_0)(r) := |\Phi[\mathcal{C}u_0](r)| \geq m, \quad \forall r \in (R_1, R_2), \quad (2.46)$$

$$J_1(u_0) := \Psi[J(u_0)] \neq 0, \quad (2.47)$$

where we have set:

$$J(u_0)(x) := \left( \mathcal{B}u_0(x) - \frac{\Phi[\mathcal{B}u_0](|x|)}{\Phi[\mathcal{C}u_0](|x|)} \mathcal{C}u_0(x) \right) \exp \left[ \int_{|x|}^{R_2} \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right], \quad \forall x \in \Omega. \quad (2.48)$$

**Remark 2.5.** If (2.1) and (2.2) hold, then according to (2.27) and (2.28) it follows that:

$$\Phi[J(u_0)](r) = \exp\left[\int_r^{R_2} \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi\right] \Phi\left(\mathcal{B}u_0 - \frac{\Phi[\mathcal{B}u_0]}{\Phi[\mathcal{C}u_0]} \mathcal{C}u_0\right)(r) = 0, \quad \forall r \in (R_1, R_2). \quad (2.49)$$

This means that operator  $\Psi$  cannot be chosen of the form  $\Psi = \Lambda\Phi$ , where  $\Lambda$  is a functional in  $L^p(R_1, R_2)^*$ , otherwise condition (2.47) would be not satisfied. In the explicit case, when  $\Phi$  and  $\Psi$  have the integral representations (1.12) and (1.13), this means that no function  $\psi$  of the form  $\psi(x) = \psi_1(|x|)\lambda(R_2|x|^{-1})$  is allowed.

**Remark 2.6.** When operators  $\Phi$  and  $\Psi$  are defined by (1.12), (1.13) conditions (2.46), (2.47) can be rewritten as:

$$\left| \int_0^\pi \sin\theta d\theta \int_0^{2\pi} \lambda(R_2 x') \mathcal{C}u_0(r x') d\varphi \right| \geq m_1, \quad \forall r \in (R_1, R_2), \quad (2.50)$$

$$\begin{aligned} & \left| \int_{R_1}^{R_2} r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} \psi(r x') \left( \mathcal{B}u_0(r x') - \frac{\int_0^\pi \sin\theta d\theta \int_0^{2\pi} \lambda(R_2 x') \mathcal{B}u_0(r x') d\varphi}{\int_0^\pi \sin\theta d\theta \int_0^{2\pi} \lambda(R_2 x') \mathcal{C}u_0(r x') d\varphi} \mathcal{C}u_0(r x') \right) \right. \\ & \quad \left. \times \exp\left[ \int_r^{R_2} \frac{\int_0^\pi \sin\theta d\theta \int_0^{2\pi} \lambda(R_2 x') \mathcal{B}u_0(\xi x') d\varphi}{\int_0^\pi \sin\theta d\theta \int_0^{2\pi} \lambda(R_2 x') \mathcal{C}u_0(\xi x') d\varphi} d\xi \right] d\varphi \right| \geq m_2 \end{aligned} \quad (2.51)$$

for some positive constants  $m_1$  and  $m_2$ .

Finally, we introduce the Banach spaces  $\mathcal{U}^{s,p}(T)$ ,  $\mathcal{U}_{H,K}^{s,p}(T)$ ,  $H, K \in \{D, N\}$ , which are defined for any  $s \in \mathbb{N} \setminus \{0\}$  by

$$\begin{cases} \mathcal{U}^{s,p}(T) = C^s([0, T]; L^p(\Omega)) \cap C^{s-1}([0, T]; W^{2,p}(\Omega)), \\ \mathcal{U}_{H,K}^{s,p}(T) = C^s([0, T]; L^p(\Omega)) \cap C^{s-1}([0, T]; W_{H,K}^{2,p}(\Omega)). \end{cases} \quad (2.52)$$

Moreover, we list some further consistency conditions:

$$(C2,D,D) \quad v_0(x) = 0, \quad \forall x \in \partial\Omega, \quad (2.53)$$

$$(C2,D,N) \quad \begin{cases} v_0(x) = 0, \\ \frac{\partial v_0}{\partial \nu}(x) = 0, \end{cases} \quad \begin{aligned} & \forall x \in \partial B(0, R_2), \\ & \forall x \in \partial B(0, R_1), \end{aligned} \quad (2.54)$$

$$(C2,N,D) \quad \begin{cases} \frac{\partial v_0}{\partial \nu}(x) = 0, \\ v_0(x) = 0, \end{cases} \quad \begin{aligned} & \forall x \in \partial B(0, R_2), \\ & \forall x \in \partial B(0, R_1), \end{aligned} \quad (2.55)$$

$$(C2,N,N) \quad \frac{\partial v_0}{\partial \nu}(x) = 0, \quad \forall x \in \partial\Omega, \quad (2.56)$$

$$\Phi[v_0](r) = D_t g_1(0, r) - \Phi[D_t u_1(0, \cdot)](r), \quad \forall r \in (R_1, R_2), \quad (2.57)$$

$$\Psi[v_0] = D_t g_2(0) - \Psi[D_t u_1(0, \cdot)], \quad (2.58)$$

where

$$v_0(x) = \mathcal{A}u_0(x) + f(0, x) - D_t u_1(0, x), \quad \forall x \in \Omega. \quad (2.59)$$

**Theorem 2.7.** *Let assumptions (1.3), (2.1) – (2.2), (2.26) – (2.31) and condition (2.3) be fulfilled. Moreover assume that the data enjoy the properties (2.33) – (2.39) and satisfy inequalities (2.46), (2.47) and consistency conditions (C1,H,K) (cf. (1.14) – (1.17)), (C2,H,K) as well as (1.18), (1.19), (2.57), (2.58).*

*Then there exists  $T^* \in (0, T]$  such that the identification problem  $P(H, K)$ ,  $H, K \in \{D, N\}$ , admits a unique solution  $(u, k) \in \mathcal{U}^{2,p}(T^*) \times C^\beta([0, T^*]; W^{1,p}(R_1, R_2))$  depending continuously on the data with respect to the norms related to the Banach spaces in (2.33) – (2.39). In the case of the specific operators  $\Phi$ ,  $\Psi$  defined by (1.12), (1.13) the previous result is still true if we assume that  $\lambda \in C^1(\partial B(0, R_2))$  and  $\psi \in C^1(\overline{\Omega})$  with  $\psi = 0$  on the part of  $\partial\Omega$  where the Dirichlet condition is possibly prescribed.*

**Lemma 2.8.** *When  $\Phi$  and  $\Psi$  are defined by (1.12) and (1.13), respectively, conditions (2.27) – (2.31) are satisfied under assumptions (2.1), (2.3) on the coefficients  $a_{i,j}$  ( $i, j = 1, 2, 3$ ) and the hypotheses that  $\lambda \in C^1(\partial B(0, R_2))$  and  $\psi \in C^1(\overline{\Omega})$  with  $\psi = 0$  on the part of  $\partial\Omega$  where the Dirichlet condition is possibly prescribed.*

*Proof.* From definitions (1.12), (1.13) it trivially follows that conditions (2.27) – (2.29) are satisfied. Hence we have only to prove that the decompositions (2.30), (2.31) hold. If the coefficients  $a_{i,j}$ ,  $i, j = 1, 2, 3$ , satisfy condition (2.3), then the second-order differential operator  $\mathcal{A}$  can be represented, in spherical co-ordinates, by operator  $\tilde{\mathcal{A}}$  defined by (2.25). Thus, taken  $w \in W_{H,K}^{2,p}(\Omega)$  with  $p \in (3, +\infty)$ , we can apply the linear functional  $\Phi$  defined in (1.12) to the right-hand side of (2.25). From the well-known formulae

$$\begin{cases} D_r = \cos\varphi \sin\theta D_{x_1} + \sin\varphi \sin\theta D_{x_2} + \cos\theta D_{x_3}, \\ D_\varphi = -r \sin\varphi \sin\theta D_{x_1} + r \cos\varphi \sin\theta D_{x_2}, \\ D_\theta = r \cos\varphi \cos\theta D_{x_1} + r \sin\varphi \sin\theta D_{x_2} - r \sin\theta D_{x_3}. \end{cases} \quad (2.60)$$

it follows that  $D_r w$ ,  $(D_\varphi w)/(r \sin\theta)$ ,  $(D_\theta w)/r$  belong to  $C^{1-3/p}([R_1, R_2] \times [0, 2\pi] \times [0, \pi])$ . Hence, differentiating under the integral sign, integrating by parts, using the periodicity with respect to  $\varphi$  of the functions  $g_j, k_j$ ,  $j = 1, 2, 3$ , defined by (2.19), (2.24) and the membership of  $\lambda$  in  $C^1(\partial B(0, R_2))$ , we obtain:

$$\Phi[\tilde{\mathcal{A}}_1 w](r) = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} \lambda(R_2 x') D_r[k_1(r) D_r w(r x')] d\varphi = \tilde{\mathcal{A}}_1 \Phi[w](r), \quad (2.61)$$

$\tilde{\mathcal{A}}_1$  being the differential operator defined in (2.32). Likewise we derive the formulae

$$\begin{aligned} \Phi\left[D_r\left(\frac{k_2(r, \varphi, \theta) D_\varphi w}{r \sin\theta}\right)\right](r) &= - \int_0^\pi d\theta \int_0^{2\pi} D_r w(r x') \frac{D_\varphi[\lambda(R_2 x') k_2(r, \varphi, \theta)]}{r} d\varphi \\ &+ \frac{1}{r} \int_0^\pi d\theta \int_0^{2\pi} w(r x') \left\{ \frac{D_\varphi[\lambda(R_2 x') k_2(r, \varphi, \theta)]}{r} - D_r D_\varphi[\lambda(R_2 x') k_2(r, \varphi, \theta)] \right\} d\varphi, \end{aligned} \quad (2.62)$$

$$\begin{aligned}
\Phi \left[ D_r \left( \frac{k_3(r, \varphi, \theta) D_\theta w}{r} \right) \right] (r) &= - \int_0^\pi d\theta \int_0^{2\pi} D_r w(r x') \frac{D_\theta [\lambda(R_2 x') k_3(r, \varphi, \theta) \sin \theta]}{r} d\varphi \\
&+ \frac{1}{r} \int_0^\pi d\theta \int_0^{2\pi} w(r x') \left\{ \frac{D_\theta [\lambda(R_2 x') k_3(r, \varphi, \theta) \sin \theta]}{r} - D_r D_\theta [\lambda(R_2 x') k_3(r, \varphi, \theta) \sin \theta] \right\} d\varphi,
\end{aligned} \tag{2.63}$$

$$\begin{aligned}
\Phi \left[ - \frac{\sin \varphi}{r \sin \theta} D_\varphi \left( f_1(r, \varphi, \theta) D_r w + \frac{f_2(r, \varphi, \theta) D_\varphi w}{r \sin \theta} + \frac{f_3(r, \varphi, \theta) D_\theta w}{r} \right) \right] (r) \\
= \int_0^\pi d\theta \int_0^{2\pi} \left[ f_1(r, \varphi, \theta) D_r w(r x') + \frac{f_2(r, \varphi, \theta) D_\varphi w(r x')}{r \sin \theta} + \frac{f_3(r, \varphi, \theta) D_\theta w(r x')}{r} \right] \\
\times \frac{D_\varphi [\lambda(R_2 x') \sin \varphi]}{r} d\varphi,
\end{aligned} \tag{2.64}$$

$$\begin{aligned}
\Phi \left[ \frac{\cos \varphi}{r \sin \theta} D_\varphi \left( g_1(r, \varphi, \theta) D_r w + \frac{g_2(r, \varphi, \theta) D_\varphi w}{r \sin \theta} + \frac{g_3(r, \varphi, \theta) D_\theta w}{r} \right) \right] (r) \\
= - \int_0^\pi d\theta \int_0^{2\pi} \left[ g_1(r, \varphi, \theta) D_r w(r x') + \frac{g_2(r, \varphi, \theta) D_\varphi w(r x')}{r \sin \theta} + \frac{g_3(r, \varphi, \theta) D_\theta w(r x')}{r} \right] \\
\times \frac{D_\varphi [\lambda(R_2 x') \cos \varphi]}{r} d\varphi,
\end{aligned} \tag{2.65}$$

$$\begin{aligned}
\Phi \left[ - \frac{\sin \theta}{r} D_\theta \left( h_1(r, \varphi, \theta) D_r w + \frac{h_2(r, \varphi, \theta) D_\varphi w}{r \sin \theta} + \frac{h_3(r, \varphi, \theta) D_\theta w}{r} \right) \right] (r) \\
= \int_0^\pi d\theta \int_0^{2\pi} \left[ h_1(r, \varphi, \theta) D_r w(r x') + \frac{h_2(r, \varphi, \theta) D_\varphi w(r x')}{r \sin \theta} + \frac{h_3(r, \varphi, \theta) D_\theta w(r x')}{r} \right] \\
\times \frac{D_\theta [\lambda(R_2 x') \sin^2 \theta]}{r} d\varphi,
\end{aligned} \tag{2.66}$$

$$\begin{aligned}
\Phi \left[ \frac{\cos \theta}{r} D_\theta \left( l_1(r, \varphi, \theta) D_r w + \frac{l_2(r, \varphi, \theta) D_\varphi w}{r \sin \theta} + \frac{l_3(r, \varphi, \theta) D_\theta w}{r} \right) \right] (r) \\
= - \int_0^\pi d\theta \int_0^{2\pi} \left[ l_1(r, \varphi, \theta) D_r w(r x') + \frac{l_2(r, \varphi, \theta) D_\varphi w(r x')}{r \sin \theta} + \frac{l_3(r, \varphi, \theta) D_\theta w(r x')}{r} \right] \\
\times \frac{D_\theta [\lambda(R_2 x') \sin 2\theta]}{2r} d\varphi.
\end{aligned} \tag{2.67}$$

After, rearranging the terms of (2.61)–(2.67) we find that for every  $w \in W_{\mathbf{H}, \mathbf{K}}^{2,p}(\Omega)$  with  $p \in (3, +\infty)$  the following equation holds:

$$\Phi[\tilde{\mathcal{A}}w](r) = \tilde{\mathcal{A}}_1 \Phi[w](r) + \Phi_1[w](r),$$

where  $\Phi_1$  is given by

$$\begin{aligned}
\Phi_1[w](r) = & \frac{1}{r} \int_0^\pi d\theta \int_0^{2\pi} w(rx') \left\{ \frac{D_\varphi[\lambda(R_2x')k_2(r, \varphi, \theta)]}{r} + \frac{D_\theta[\lambda(R_2x')k_3(r, \varphi, \theta) \sin\theta]}{r} \right. \\
& - D_r D_\varphi[\lambda(R_2x')k_2(r, \varphi, \theta)] - D_r D_\theta[\lambda(R_2x')k_3(r, \varphi, \theta) \sin\theta] \Big\} d\varphi + \int_0^\pi d\theta \int_0^{2\pi} D_r w(rx') \\
& \times \left\{ \frac{f_1(r, \varphi, \theta) D_\varphi[\lambda(R_2x') \sin\varphi]}{r} - \frac{g_1(r, \varphi, \theta) D_\varphi[\lambda(R_2x') \cos\varphi]}{r} + \frac{h_1(r, \varphi, \theta) D_\theta[\lambda(R_2x') \sin^2\theta]}{r} \right. \\
& - \frac{l_1(r, \varphi, \theta) D_\theta[\lambda(R_2x') \sin 2\theta]}{2r} - \frac{D_\varphi[\lambda(R_2x')k_2(r, \varphi, \theta)]}{r} - \frac{D_\theta[\lambda(R_2x')k_3(r, \varphi, \theta) \sin\theta]}{r} \Big\} d\varphi \\
& + \int_0^\pi d\theta \int_0^{2\pi} \frac{D_\varphi w(rx')}{r \sin\theta} \left\{ \frac{f_2(r, \varphi, \theta) D_\varphi[\lambda(R_2x') \sin\varphi]}{r} - \frac{g_2(r, \varphi, \theta) D_\varphi[\lambda(R_2x') \cos\varphi]}{r} \right. \\
& + \frac{h_2(r, \varphi, \theta) D_\theta[\lambda(R_2x') \sin^2\theta]}{r} - \frac{l_2(r, \varphi, \theta) D_\theta[\lambda(R_2x') \sin 2\theta]}{2r} \Big\} d\varphi \\
& + \int_0^\pi d\theta \int_0^{2\pi} \frac{D_\theta w(rx')}{r} \left\{ \frac{f_3(r, \varphi, \theta) D_\varphi[\lambda(R_2x') \sin\varphi]}{r} - \frac{g_3(r, \varphi, \theta) D_\varphi[\lambda(R_2x') \cos\varphi]}{r} \right. \\
& + \frac{h_3(r, \varphi, \theta) D_\theta[\lambda(R_2x') \sin^2\theta]}{r} - \frac{l_3(r, \varphi, \theta) D_\theta[\lambda(R_2x') \sin 2\theta]}{2r} \Big\} d\varphi. \tag{2.68}
\end{aligned}$$

We now prove that  $\Phi_1$  belongs to  $\mathcal{L}(W^{1,p}(\Omega); L^p(R_1, R_2))$ , i.e. it is bounded from  $W^{1,p}(\Omega)$  to  $L^p(R_1, R_2)$ .

By virtue of assumption (2.1) on the coefficients  $a_{i,j}$ ,  $i, j = 1, 2, 3$ , from (2.18) – (2.20), (2.24) we deduce that  $f_j, g_j, h_j, k_j, l_j$  belong to  $W^{2,\infty}((R_1, R_2) \times (0, 2\pi) \times (0, \pi))$ ,  $j = 1, 2, 3$ . Then, since  $\lambda \in C^1(\partial B(0, R_2))$  and  $0 < R_2^{-1} < r^{-1} < R_1^{-1}$ , using formulae (2.60) we can prove that the following functions

$$\begin{aligned}
& D_r D_\theta[\lambda(R_2x')k_3(r, \varphi, \theta) \sin\theta], \quad D_r D_\varphi[\lambda(R_2x')k_2(r, \varphi, \theta)], \\
& \frac{D_\varphi[\lambda(R_2x')k_2(r, \varphi, \theta)]}{r}, \quad \frac{D_\theta[\lambda(R_2x')k_3(r, \varphi, \theta) \sin\theta]}{r}, \quad \frac{f_j(r, \varphi, \theta) D_\varphi[\lambda(R_2x') \sin\varphi]}{r}, \\
& \frac{g_j(r, \varphi, \theta) D_\varphi[\lambda(R_2x') \cos\varphi]}{r}, \quad \frac{h_j(r, \varphi, \theta) D_\theta[\lambda(R_2x') \sin^2\theta]}{r}, \quad \frac{l_j(r, \varphi, \theta) D_\theta[\lambda(R_2x') \sin 2\theta]}{2r}
\end{aligned}$$

belong to  $L^\infty((R_1, R_2) \times (0, 2\pi) \times (0, \pi))$  and their  $L^\infty$ -norms are bounded from above by  $C\|\lambda\|_{C^1(\partial B(0, R_2))}$ ,  $C$  being a positive constant depending on  $R_1, R_2, \max_{i,j=1,2,3} \|a_{i,j}\|_{W^{2,\infty}(\Omega)}$ , only.

Observe now that for any pair of functions  $f \in C(\overline{\Omega})$  and  $v \in L^p(\Omega)$  we have

$$\int_0^\pi d\theta \int_0^{2\pi} |v(rx')f(rx')| d\varphi \leq \|f\|_{C(\overline{\Omega})} \left[ \int_0^\pi \sin\theta d\theta \int_0^{2\pi} |v(rx')|^p d\varphi \right]^{1/p} \left[ 2\pi \int_0^\pi (\sin\theta)^{-\frac{1}{p-1}} d\theta \right]^{p/(p-1)} \tag{2.69}$$

Since the right-hand side in (2.69) is in  $L^p(R_1, R_2)$  when  $p \in (3, +\infty)$ , applying Hölder's inequality to the right-hand side of (2.68) we find  $\|\Phi_1[w]\|_{L^p(R_1, R_2)} \leq C\|w\|_{W^{1,p}(\Omega)}$ . Consequently decomposition (2.30) holds.

Let now  $\Psi \in L^p(\Omega)^*$  be the functional defined in (1.13). Analogously to what we have done for  $\Phi$ , we apply  $\Psi$  to both sides in (2.25). Performing computations similar to those made above and using the assumption that  $\psi|_\Gamma = 0$ , when the Dirichlet condition is prescribed on  $\Gamma \subset \partial\Omega$ , we obtain the equation:

$$\Psi[\tilde{\mathcal{A}}w] = \Psi_1[w], \quad \forall w \in W_{H,K}^{2,p}(\Omega),$$

where

$$\begin{aligned} \Psi_1[w] = & - \int_{R_1}^{R_2} r dr \int_0^\pi d\theta \int_0^{2\pi} D_r w(r x') \left\{ \frac{k_1(r) D_r[r^2 \psi(r x')] \sin \theta}{r} - f_1(r, \varphi, \theta) D_\varphi[\psi(r x') \sin \varphi] \right. \\ & + g_1(r, \varphi, \theta) D_\varphi[\psi(r x') \cos \varphi] - h_1(r, \varphi, \theta) D_\theta[\psi(r x') \sin^2 \theta] + \left. \frac{l_1(r, \varphi, \theta) D_\theta[\psi(r x') \sin 2\theta]}{2} \right\} d\varphi \\ & - \int_{R_1}^{R_2} r dr \int_0^\pi d\theta \int_0^{2\pi} \frac{D_\varphi w(r x')}{r \sin \theta} \left\{ \frac{k_2(r, \varphi, \theta) D_r[r^2 \psi(r x')] \sin \theta}{r} - f_2(r, \varphi, \theta) D_\varphi[\psi(r x') \sin \varphi] \right. \\ & + g_2(r, \varphi, \theta) D_\varphi[\psi(r x') \cos \varphi] - h_2(r, \varphi, \theta) D_\theta[\psi(r x') \sin^2 \theta] + \left. \frac{l_2(r, \varphi, \theta) D_\theta[\psi(r x') \sin 2\theta]}{2} \right\} d\varphi \\ & - \int_{R_1}^{R_2} r dr \int_0^\pi d\theta \int_0^{2\pi} \frac{D_\theta w(r x')}{r} \left\{ \frac{k_3(r, \varphi, \theta) D_r[r^2 \psi(r x')] \sin \theta}{r} - f_3(r, \varphi, \theta) D_\varphi[\psi(r x') \sin \varphi] \right. \\ & + g_3(r, \varphi, \theta) D_\varphi[\psi(r x') \cos \varphi] - h_3(r, \varphi, \theta) D_\theta[\psi(r x') \sin^2 \theta] + \left. \frac{l_3(r, \varphi, \theta) D_\theta[\psi(r x') \sin 2\theta]}{2} \right\} d\varphi. \end{aligned} \quad (2.70)$$

Since the functions  $f_j, g_j, h_j, k_j, l_j$ ,  $j = 1, 2, 3$ , belong to  $W^{2,\infty}(\Omega)$  and  $\psi \in C^1(\overline{\Omega})$ , using an estimate similar to (2.69), it easily follows that  $\Psi_1 \in W_{H,K}^{1,p}(\Omega)^*$ . Hence decomposition (2.31) also holds. This completes the proof.  $\square$

**Remark 2.9.** The reason why we have restricted ourselves to investigating the identification problem P(H,K) in the spherical corona  $\Omega = \{x \in \mathbb{R}^3 : R_1 < |x| < R_2\}$ ,  $0 < R_1 < R_2$ , instead of the simpler ball  $\Omega_1 = \{x \in \mathbb{R}^3 : |x| < R\}$ ,  $R > 0$ , is due to the representation (2.68) of the functional  $\Phi_1$ . Indeed, the function appearing in the right-hand side of (2.68) might not belong to  $L^p(0, \varepsilon)$  for any  $\varepsilon \in (0, R_2)$  when dealing with general coefficients  $a_{i,j} \in W^{2,\infty}(\Omega_1)$ . This would imply  $\Phi_1 \notin \mathcal{L}(W^{1,p}(\Omega_1); L^p(0, R))$  and would prevent us from applying known abstract results.

### 3 An equivalence result in the concrete case

In this section we prove an equivalence theorem which will be the starting point to reduce our problem to the same abstract integral fixed-point system studied in [CL].

Let us suppose that  $(u, k) \in \mathcal{U}^{2,p}(T) \times C^\beta([0, T]; W^{1,p}(R_1, R_2))$  is a solution to the identification problem P(H,K). Let us now introduce the following new unknown function

$$\begin{aligned} v(t, x) &= D_t u(t, x) - D_t u_1(t, x) \iff \\ u(t, x) &= u_1(t, x) - u_1(0, x) + u_0(x) + \int_0^t v(s, x) ds. \end{aligned} \quad (3.1)$$

Then from (1.1) it follows that the pair  $(v, k) \in \mathcal{U}_{H,K}^{1,p}(T) \times C^\beta([0, T]; W^{1,p}(R_1, R_2))$  solves the identification problem

$$\begin{aligned} D_t v(t, x) &= \mathcal{A}v(t, x) + \int_0^t k(t-s, |x|) [\mathcal{B}v(s, x) + \mathcal{B}D_t u_1(s, x)] ds + k(t, |x|) \mathcal{B}u_0(x) \\ &\quad + \int_0^t D_{|x|} k(t-s, |x|) [\mathcal{C}v(s, x) + \mathcal{C}D_t u_1(s, x)] ds + D_{|x|} k(t, |x|) \mathcal{C}u_0(x) \\ &\quad + \mathcal{A}D_t u_1(t, x) - D_t^2 u_1(t, x) + D_t f(t, x), \quad \forall (t, x) \in [0, T] \times \Omega, \end{aligned} \quad (3.2)$$

$$v(0, x) = \mathcal{A}u_0(x) + f(0, x) - D_t u_1(0, x) =: v_0(x), \quad \forall x \in \Omega, \quad (3.3)$$

$$v \text{ satisfies the homogeneous boundary conditions (H,K),} \quad (3.4)$$

$$\Phi[v(t, \cdot)](r) = D_t g_1(t, r) - \Phi[D_t u_1(t, \cdot)](r), \quad \forall (t, r) \in [0, T] \times (R_1, R_2), \quad (3.5)$$

$$\Psi[v(t, \cdot)] = D_t g_2(t) - \Psi[D_t u_1(t, \cdot)], \quad \forall t \in [0, T]. \quad (3.6)$$

The consistency conditions related to problem (3.2)–(3.6) can be deduced as in section 1 with  $(u_0, u_1, g_1, g_2)$  replaced by  $(v_0, 0, D_t g_1 - \Phi[D_t u_1], D_t g_2 - \Psi[D_t u_1])$  and they are explicitly given by (2.53)–(2.58).

Using assumptions (2.27)–(2.31) and applying the functionals  $\Phi, \Psi$  to both sides of (3.2), it easy to check that the radial kernel  $k$  satisfies the two following equations:

$$\begin{aligned} D_r k(t, r) \Phi[\mathcal{C}u_0](r) + k(t, r) \Phi[\mathcal{B}u_0](r) &= N_1^0(u_1, g_1, f)(t, r) + \Phi[\tilde{N}_1(v, k)(t, \cdot)](r) \\ &\quad - \Phi_1[v(t, \cdot)](r), \quad \forall (t, r) \in [0, T] \times (R_1, R_2), \end{aligned} \quad (3.7)$$

$$\begin{aligned} \Psi[D_r k(t, \cdot) \mathcal{C}u_0 + k(t, \cdot) \mathcal{B}u_0] &= N_2^0(u_1, g_2, f)(t) + \Psi[\tilde{N}_1(v, k)(t, \cdot)] - \Psi_1[v(t, \cdot)], \\ &\quad \forall t \in [0, T], \end{aligned} \quad (3.8)$$

where the operators  $\tilde{N}_1, N_1^0, N_2^0$  are defined, respectively, by

$$\begin{aligned} \tilde{N}_1(v, k)(t, r) &= - \int_0^t k(t-s, |x|) [\mathcal{B}v(s, x) + \mathcal{B}D_t u_1(s, x)] ds \\ &\quad - \int_0^t D_{|x|} k(t-s, |x|) [\mathcal{C}v(s, x) + \mathcal{C}D_t u_1(s, x)] ds, \quad \forall (t, x) \in [0, T] \times \Omega, \end{aligned} \quad (3.9)$$

$$\begin{aligned} N_1^0(u_1, g_1, f)(t, r) &= D_t^2 g_1(t, r) - D_t \tilde{\mathcal{A}}_1 g_1(t, r) - \Phi_1[D_t u_1(t, \cdot)](r) - \Phi[D_t f(t, \cdot)](r), \\ &\quad \forall (t, r) \in [0, T] \times (R_1, R_2), \end{aligned} \quad (3.10)$$

$$N_2^0(u_1, g_2, f)(t) = D_t^2 g_2(t) - \Psi_1[D_t u_1(t, \cdot)] - \Psi[D_t f(t, \cdot)], \quad \forall t \in [0, T]. \quad (3.11)$$

**Remark 3.1.** It can be easily checked that if  $(v, k) \in \mathcal{U}_{H,K}^{1,p}(T) \times C^\beta([0, T]; W^{1,p}(R_1, R_2))$  solves the identification problem (3.2)–(3.8) then, taking advantage of the consistency conditions (2.53)–(2.58), the function  $u \in \mathcal{U}^{2,p}(T)$  defined in (3.1) is a solution to the problem P(H,K).

From (3.7), (3.8) it turns out that the initial value  $k(0, \cdot)$  must satisfy the following equations:

$$D_r k(0, r) \Phi[\mathcal{C}u_0](r) + k(0, r) \Phi[\mathcal{B}u_0](r) = N_1^0(u_1, g_1, f)(0, r) - \Phi_1[v_0](r), \quad \forall r \in (R_1, R_2), \quad (3.12)$$

$$\Psi[D_r k(0, \cdot) \mathcal{C}u_0 + k(0, \cdot) \mathcal{B}u_0] = N_2^0(u_1, g_2, f)(0) - \Psi_1[v_0]. \quad (3.13)$$

Let

$$\tilde{l}_1(r) := N_1^0(u_1, g_1, f)(0, r) - \Phi_1[v_0](r), \quad \forall r \in (R_1, R_2). \quad (3.14)$$

Then using condition (2.46) and integrating the first-order differential equation (3.12) we obtain the following general integral depending on an arbitrary constant  $C$ :

$$k(0, r) = C \exp \left[ \int_r^{R_2} \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right] + \int_{R_2}^r \exp \left[ \int_r^\eta \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right] \frac{\tilde{l}_1(\eta)}{\Phi[\mathcal{C}u_0](\eta)} d\eta. \quad (3.15)$$

Substituting this representation of  $k(0, \cdot)$  into (3.13), we can compute the constant  $C$ :

$$C = [J_1(u_0)]^{-1} \left\{ \Psi[\tilde{l}_2] + N_2^0(u_1, g_2, f)(0) - \Psi_1[v_0] \right\}, \quad (3.16)$$

where  $J_1(u_0)$  and  $\tilde{l}_2$  are defined, respectively, by (2.47) and the following formula:

$$\begin{aligned} \tilde{l}_2(x) := & \mathcal{C}u_0(x) \left\{ \frac{\tilde{l}_1(|x|)}{\Phi[\mathcal{C}u_0](|x|)} - \frac{\Phi[\mathcal{B}u_0](|x|)}{\Phi[\mathcal{C}u_0](|x|)} \int_{R_2}^{|x|} \exp \left[ \int_{|x|}^\eta \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right] \frac{\tilde{l}_1(\eta)}{\Phi[\mathcal{C}u_0](\eta)} d\eta \right\} \\ & + \mathcal{B}u_0(x) \int_{R_2}^{|x|} \exp \left[ \int_{|x|}^\eta \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right] \frac{\tilde{l}_1(\eta)}{\Phi[\mathcal{C}u_0](\eta)} d\eta, \quad \forall x \in \Omega. \end{aligned} \quad (3.17)$$

Then, substituting (3.16) into (3.15), we find that the initial value  $k(0, \cdot)$  is given by

$$\begin{aligned} k(0, r) = & [J_1(u_0)]^{-1} \left\{ \Psi[\tilde{l}_2] + N_2^0(u_1, g_2, f)(0) - \Psi_1[v_0] \right\} \exp \left[ \int_r^{R_2} \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right] \\ & + \int_{R_2}^r \exp \left[ \int_r^\eta \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right] \frac{\tilde{l}_1(\eta)}{\Phi[\mathcal{C}u_0](\eta)} d\eta := k_0(r), \quad \forall r \in (R_1, R_2). \end{aligned} \quad (3.18)$$

Now we introduce the two new unknown functions

$$h(t) = k(t, R_2), \quad q(t, r) = D_r k(t, r), \quad \forall (t, r) \in [0, T] \times (R_1, R_2). \quad (3.19)$$

and express  $k$  in terms of  $h$  and  $q$ :

$$k(t, r) = h(t) - \int_r^{R_2} q(t, \xi) d\xi := h(t) - Eq(t, r), \quad \forall (t, r) \in [0, T] \times (R_1, R_2). \quad (3.20)$$

Of course, (3.19) and (3.20) imply the initial conditions

$$h(0) = k_0(R_2), \quad q(0, r) = k'_0(r), \quad \forall r \in [R_1, R_2]. \quad (3.21)$$

Using (3.20), we solve (3.7), (3.8) for the pair  $(h, q)$ . From definition (3.9) we deduce the following representation for operator  $\tilde{N}_1$ :

$$\begin{aligned} \tilde{N}_1(v, h - Eq)(t, |x|) &:= - \int_0^t [h(t-s) - Eq(t-s, |x|)] [\mathcal{B}v(s, x) + \mathcal{B}D_t u_1(s, x)] ds \\ &\quad - \int_0^t q(t-s, |x|) [\mathcal{C}v(s, x) + \mathcal{C}D_t u_1(s, x)] ds \\ &:= N_1(v, h, q)(t, |x|), \quad \forall (t, x) \in [0, T] \times \Omega. \end{aligned} \quad (3.22)$$

Moreover system (3.7), (3.8) changes into

$$\begin{aligned} q(t, r)\Phi[\mathcal{C}u_0](r) - Eq(t, r)\Phi[\mathcal{B}u_0](r) &= N_1^0(u_1, g_1, f)(t, r) - h(t)\Phi[\mathcal{B}u_0](r) - \Phi_1[v(t, \cdot)](r) \\ &\quad + \Phi[N_1(v, h, q)(t, \cdot)](r), \quad \forall (t, r) \in [0, T] \times (R_1, R_2), \end{aligned} \quad (3.23)$$

$$\begin{aligned} \Psi[q(t, \cdot)\mathcal{C}u_0 + [h(t) - Eq(t, \cdot)]\mathcal{B}u_0] &= N_2^0(u_1, g_2, f)(t) + \Psi[N_1(v, h, q)(t, \cdot)] - \Psi_1[v(t, \cdot)], \\ &\quad \forall t \in [0, T]. \end{aligned} \quad (3.24)$$

First we consider the integral equation

$$q(t, r)\Phi[\mathcal{C}u_0](r) - Eq(t, r)\Phi[\mathcal{B}u_0](r) = g(t, x), \quad \forall (t, r) \in [0, T] \times (R_1, R_2), \quad (3.25)$$

where  $g \in L^1((0, T) \times (R_1, R_2))$  is an arbitrary given function.

Since  $u_0$  satisfies condition (2.46) and (3.20) implies  $Eq(t, R_2) = 0$  for all  $t \in [0, T]$  and  $D_r Eq(t, r) = -q(t, r)$  for all  $(t, r) \in [0, T] \times (R_1, R_2)$ , the solution to the differential equation (3.25) is given by

$$Eq(t, r) = Lg(t, r), \quad (3.26)$$

where operator  $L$  is defined by the formula

$$Lg(t, r) := \int_r^{R_2} \exp \left[ \int_r^\eta \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right] \frac{g(t, \eta)}{\Phi[\mathcal{C}u_0](\eta)} d\eta. \quad (3.27)$$

Hence, using (3.26) and the relation  $D_r Eq(t, r) = -q(t, r)$ , from (3.27) we obtain the following representation formula for  $q$ :

$$q(t, r) = \frac{1}{\Phi[\mathcal{C}u_0](r)} [I + \Phi[\mathcal{B}u_0](r)L] g(t, r), \quad \forall (t, r) \in [0, T] \times (R_1, R_2), \quad (3.28)$$

where  $I$  denotes the identity operator.

From (3.23) we get:

$$g(t, r) = N_1^0(u_1, g_1, f)(t, r) - h(t)\Phi[\mathcal{B}u_0](r) + \Phi[N_1(v, h, q)(t, \cdot)](r) - \Phi_1[v(t, \cdot)](r).$$

Therefore substituting into (3.28) we find:

$$q(t, r) = -h(t) \frac{\Phi[\mathcal{B}u_0](r)}{\Phi[\mathcal{C}u_0](r)} [1 + L\Phi[\mathcal{B}u_0](r)] + N_3^0(u_0, u_1, g_1, f)(t, r) + N_2(v, h, q)(t, r),$$

$$\forall (t, r) \in [0, T] \times (R_1, R_2), \quad (3.29)$$

where we have set:

$$N_2(v, h, q)(t, r) = \frac{1}{\Phi[\mathcal{C}u_0](r)} [I + \Phi[\mathcal{B}u_0](r)L] \{ \Phi[N_1(v, h, q)(t, \cdot)](r) - \Phi_1[v(t, \cdot)](r) \}$$

$$=: J_3(u_0)(r) \{ \Phi[N_1(v, h, q)(t, \cdot)](r) - \Phi_1[v(t, \cdot)](r) \}, \quad (3.30)$$

$$N_3^0(u_0, u_1, g_1, f)(t, r) = \frac{1}{\Phi[\mathcal{C}u_0](r)} [I + \Phi[\mathcal{B}u_0](r)L] N_1^0(u_1, g_1, f)(t, r)$$

$$=: J_3(u_0)(r) N_1^0(u_1, g_1, f)(t, r). \quad (3.31)$$

Observing that (3.27) implies

$$1 + L\Phi[\mathcal{B}u_0](r) = \exp \left[ \int_r^{R_2} \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right], \quad \forall r \in (R_1, R_2), \quad (3.32)$$

and substituting this expression into (3.29), from (3.24) it is easy to check that  $h$  solves the following equation:

$$h(t)J_1(u_0) = N_0(u_0, u_1, g_1, g_2, f)(t) + \Psi[N_1(v, h, q)(t, \cdot)] - \Psi[N_2(v, h, q)(t, \cdot)\mathcal{C}u_0]$$

$$+ \Psi[E(N_2(v, h, q)(t, \cdot))\mathcal{B}u_0] - \Psi_1[v(t, \cdot)], \quad \forall t \in [0, T], \quad (3.33)$$

where  $J_1(u_0)$  and  $N_0(u_0, u_1, g_1, g_2, f)$  are defined, respectively, by (2.47) and

$$N_0(u_0, u_1, g_1, g_2, f)(t) := N_2^0(u_1, g_2, f)(t) - \Psi[N_3^0(u_0, u_1, g_1, f)(t, \cdot)\mathcal{C}u_0]$$

$$- \Psi[E(N_3^0(u_0, u_1, g_1, f)(t, \cdot))\mathcal{B}u_0], \quad \forall t \in [0, T]. \quad (3.34)$$

Hence, from (3.33) and (2.47) we conclude that  $h$  solves the following fixed-point equation:

$$h(t) = h_0(t) + N_3(v, h, q)(t), \quad \forall t \in [0, T], \quad (3.35)$$

where we have set:

$$h_0(t) := [J_1(u_0)]^{-1} N_0(u_0, u_1, g_1, g_2, f)(t), \quad (3.36)$$

$$N_3(v, h, q)(t) := [J_1(u_0)]^{-1} \left\{ \Psi[N_1(v, h, q)(t, \cdot)] - \Psi[N_2(v, h, q)(t, \cdot)\mathcal{C}u_0] \right.$$

$$\left. + \Psi[E(N_2(v, h, q)(t, \cdot))\mathcal{B}u_0] - \Psi_1[v(t, \cdot)] \right\}. \quad (3.37)$$

So, using again (3.32) and replacing the right-hand side of (3.35) into (3.29), we conclude that  $q$  satisfies the following fixed-point equation

$$q(t, r) = q_0(t, r) + J_2(u_0)(r)N_3(v, h, q)(t) + N_2(v, h, q)(t, r),$$

$$\forall (t, r) \in [0, T] \times (R_1, R_2), \quad (3.38)$$

where

$$J_2(u_0)(r) = -\frac{\Phi[\mathcal{B}u_0](r)}{\Phi[\mathcal{C}u_0](r)} \exp \left[ \int_r^{R_2} \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} d\xi \right], \quad \forall r \in (R_1, R_2), \quad (3.39)$$

and

$$q_0(t, r) := J_2(u_0)(r)h_0(t) + N_3^0(u_0, u_1, g_1, f)(t, r), \quad \forall (t, r) \in [0, T] \times (R_1, R_2). \quad (3.40)$$

We have thus shown that the pair  $(h, q)$  solves the fixed-point system (3.35), (3.38).

We can summarize the result of this section in the following equivalence theorem.

**Theorem 3.2.** *The pair  $(u, k) \in \mathcal{U}^{2,p}(T) \times C^\beta([0, T]; W^{1,p}(R_1, R_2))$  is a solution to the identification problem  $P(H, K)$ ,  $H, K \in \{D, N\}$ , if and only if the triplet  $(v, h, q)$  defined by (3.1) and (3.19) belongs to  $\mathcal{U}_{H,K}^{1,p}(T) \times C^\beta([0, T]; \mathbb{R}) \times C^\beta([0, T]; L^p(R_1, R_2))$  and solves problem (3.2)–(3.6), (3.35), (3.38).*

## 4 An abstract formulation of problem (3.2)–(3.6), (3.35), (3.38).

Starting from the result of the previous section, we can reformulate our identification problem in a Banach space framework. Let  $A : \mathcal{D}(A) \subset X \rightarrow X$  be a linear closed operator satisfying the following assumptions:

- (H1) *there exists  $\zeta \in (\pi/2, \pi)$  such that the resolvent set of  $A$  contains 0 and the open sector  $\Sigma_\zeta = \{\mu \in \mathbb{C} : |\arg \mu| < \zeta\}$ ;*
- (H2) *there exists  $M > 0$  such that  $\|(\mu I - A)^{-1}\|_{\mathcal{L}(X)} \leq M|\mu|^{-1}$  for every  $\mu \in \Sigma_\zeta$ .*
- (H3)  *$X_1$  and  $X_2$  are Banach spaces such that  $\mathcal{D}(A) = X_2 \hookrightarrow X_1 \hookrightarrow X$ . Moreover,  $\mu \rightarrow (\mu I - A)^{-1}$  belongs to  $\mathcal{L}(X; X_1)$  and satisfies the estimate  $\|(\mu I - A)^{-1}\|_{\mathcal{L}(X; X_1)} \leq M|\mu|^{-1/2}$  for every  $\mu \in \Sigma_\zeta$ .*

Here  $\mathcal{L}(Z_1; Z_2)$  denotes, for any pair of Banach spaces  $Z_1$  and  $Z_2$ , the Banach space of all bounded linear operators from  $Z_1$  into  $Z_2$  equipped with the uniform-norm. In particular we set  $\mathcal{L}(X) = \mathcal{L}(X; X)$ .

By virtue of assumptions (H1), (H2) we can define the analytic semigroup  $\{e^{tA}\}_{t \geq 0}$  of bounded linear operators in  $\mathcal{L}(X)$  generated by  $A$ . As is well-known, there exist positive constants  $\tilde{c}_k(\zeta)$  ( $k \in \mathbb{N}$ ) such that

$$\|A^k e^{tA}\|_{\mathcal{L}(X)} \leq \tilde{c}_k(\zeta) M t^{-k}, \quad \forall t \in \mathbb{R}_+, \quad \forall k \in \mathbb{N}.$$

After endowing  $\mathcal{D}(A)$  with the graph-norm, we can define the following family of interpolation spaces  $\mathcal{D}_A(\beta, p)$ ,  $\beta \in (0, 1)$ ,  $p \in [1, +\infty]$ , which are intermediate between  $\mathcal{D}(A)$  and  $X$ :

$$\mathcal{D}_A(\beta, p) = \left\{ x \in X : |x|_{\mathcal{D}_A(\beta, p)} < +\infty \right\}, \quad \text{if } p \in [1, +\infty], \quad (4.1)$$

where

$$|x|_{\mathcal{D}_A(\beta,p)} = \begin{cases} \left( \int_0^{+\infty} t^{(1-\beta)p-1} \|Ae^{tA}x\|_X^p dt \right)^{1/p}, & \text{if } p \in [1, +\infty), \\ \sup_{0 < t \leq 1} (t^{1-\beta} \|Ae^{tA}x\|_X), & \text{if } p = \infty. \end{cases} \quad (4.2)$$

They are well defined by virtue of assumption (H1). Moreover, we set

$$\mathcal{D}_A(1 + \beta, p) = \{x \in \mathcal{D}(A) : Ax \in \mathcal{D}_A(\beta, p)\}. \quad (4.3)$$

Consequently,  $\mathcal{D}_A(n + \beta, p)$ ,  $n \in \mathbb{N}, \beta \in (0, 1), p \in [1, +\infty]$ , turns out to be a Banach space when equipped with the norm

$$\|x\|_{\mathcal{D}_A(n+\beta,p)} = \sum_{j=0}^n \|A^j x\|_X + |A^n x|_{\mathcal{D}_A(\beta,p)}. \quad (4.4)$$

In order to reformulate in an abstract form our identification problem (3.2)–(3.6), (3.35), (3.38) we need the following assumptions involving spaces, operators and data:

- (H4)  $Y$  and  $Y_1$  are Banach spaces such that  $Y_1 \hookrightarrow Y$ ;
- (H5)  $B : \mathcal{D}(B) \subset X \rightarrow X$  is a linear closed operator such that  $X_2 \subset \mathcal{D}(B)$ ;
- (H6)  $C : \mathcal{D}(C) := X_1 \subset X \rightarrow X$  is a linear closed operator;
- (H7)  $E \in \mathcal{L}(Y; Y_1)$ ,  $\Phi \in \mathcal{L}(X; Y)$ ,  $\Psi \in X^*$ ,  $\Phi_1 \in \mathcal{L}(X_1; Y)$ ,  $\Psi_1 \in X_1^*$ ;
- (H8)  $\mathcal{M}$  is a continuous bilinear operator from  $Y \times \tilde{X}_1$  to  $X$  and from  $Y_1 \times X$  to  $X$ , where  $X_1 \hookrightarrow \tilde{X}_1$ ;
- (H9)  $J_1 : X_2 \rightarrow \mathbb{R}$ ,  $J_2 : X_2 \rightarrow Y$ ,  $J_3 : X_2 \rightarrow \mathcal{L}(Y)$  are three prescribed (non-linear) operators;
- (H10)  $u_0, v_0 \in X_2$ ,  $Cu_0 \in X_1$ ,  $J_1(u_0) \neq 0$ ,  $Bu_0 \in \mathcal{D}_A(\delta, +\infty)$ ,  $\delta \in (\beta, 1/2)$ ;
- (H11)  $q_0 \in C^\beta([0, T]; Y)$ ,  $h_0 \in C^\beta([0, T])$ ;
- (H12)  $z_0 \in C^\beta([0, T]; X)$ ,  $z_1 \in C^\beta([0, T]; \tilde{X}_1)$ ,  $z_2 \in C^\beta([0, T]; X)$ ;
- (H13)  $Av_0 + \mathcal{M}(\tilde{q}_0, Cu_0) + \tilde{h}_0 Bu_0 - \mathcal{M}(E\tilde{q}_0, Bu_0) + z_2(0, \cdot) \in \mathcal{D}_A(\beta, +\infty)$ ;

where  $\tilde{h}_0$  and  $\tilde{q}_0$  are defined in the following Remark 4.2.

We can now reformulate our problem: *determine a function  $v \in C^1([0, T]; X) \cap C([0, T]; X_2)$  such that*

$$\begin{aligned} v'(t) = & [\lambda_0 I + A]v(t) + \int_0^t h(t-s)[Bv(s) + z_0(s)]ds - \int_0^t \mathcal{M}(Eq(t-s), Bv(s) + z_0(s))ds \\ & + \int_0^t \mathcal{M}(q(t-s), Cv(s) + z_1(s))ds + \mathcal{M}(q(t), Cu_0) + h(t)Bu_0 \\ & - \mathcal{M}(Eq(t), Bu_0) + z_2(t), \quad \forall t \in [0, T], \end{aligned} \quad (4.5)$$

$$v(0) = v_0. \quad (4.6)$$

**Remark 4.1.** In the explicit case (3.2), (3.6) we have  $A = \mathcal{A} - \lambda_0 I$ , with a large enough positive  $\lambda_0$ , and the functions  $z_0, z_1, z_2$  defined by

$$z_0 = D_t \mathcal{B}u_1, \quad z_1 = D_t \mathcal{C}u_1, \quad z_2 = D_t \mathcal{A}u_1 - D_t^2 u_1 + D_t f, \quad (4.7)$$

whereas  $v_0, h_0, q_0$  are defined, respectively, via the formulae (3.3), (3.36), (3.40).

Let us now introduce the following unknown function  $w$  related to  $v$  by

$$w = Av \iff v = A^{-1}w. \quad (4.8)$$

Applying  $A$  to the Volterra operator equation equivalent to problem (4.5), (4.6) and using (4.8), we can easily obtain the following equation for  $w$ :

$$\begin{aligned} w(t) = & Ae^{tA}v_0 + \lambda_0 \int_0^t e^{(t-s)A}w(s)ds + A \int_0^t e^{(t-s)A}z_2(s)ds \\ & + A \int_0^t e^{(t-s)A}ds \int_0^s h(s-\sigma)[BA^{-1}w(\sigma) + z_0(\sigma)]d\sigma \\ & - A \int_0^t e^{(t-s)A}ds \int_0^s \mathcal{M}(Eq(s-\sigma), BA^{-1}w(\sigma) + z_0(\sigma))d\sigma \\ & + A \int_0^t e^{(t-s)A}ds \int_0^s \mathcal{M}(q(s-\sigma), CA^{-1}w(\sigma) + z_1(\sigma))d\sigma \\ & + A \int_0^t h(s)e^{(t-s)A}Bu_0ds - A \int_0^t e^{(t-s)A}\mathcal{M}(Eq(s), Bu_0)ds \\ & + A \int_0^t e^{(t-s)A}\mathcal{M}(q(s), Cu_0)ds, \quad \forall t \in [0, T]. \end{aligned} \quad (4.9)$$

Denoting by  $\mathcal{K}$  the convolution operator

$$\mathcal{K}(f, g) := \int_0^t \mathcal{M}(f(t-s), g(s))ds, \quad (4.10)$$

which maps  $C^\beta([0, T]; Y_1) \times C([0, T]; X)$  and  $C^\beta([0, T]; Y) \times C([0, T]; X_1)$  into  $C^\beta([0, T]; X)$  (cf. [CL], section 4), we can rewrite equation (4.9) in the more compact way

$$w = w_0 + R_1(w, h, q) + S_1(q), \quad (4.11)$$

where we have set

$$\begin{aligned} R_1(w, h, q) := & \lambda_0(e^{tA} * w) + A[h * e^{tA} * (BA^{-1}w + z_0)] - A[e^{tA} * \mathcal{K}(Eq, BA^{-1}w + z_0)] \\ & + A[e^{tA} * \mathcal{K}(q, CA^{-1}w + z_1)] + A[e^{tA} * hBu_0] - A[e^{tA} * \mathcal{M}(Eq, Bu_0)], \end{aligned} \quad (4.12)$$

and

$$S_1(q) := A[e^{tA} * \mathcal{M}(q, Cu_0)], \quad (4.13)$$

$$w_0 := Ae^{tA}v_0 + A(e^{tA} * z_2). \quad (4.14)$$

Hence, applying operator  $A^{-1}$  to both hand sides of (4.11), from (4.12) – (4.14) we get

$$\begin{aligned}
v &= A^{-1}w_0 + \lambda_0(e^{tA} * v) + e^{tA} * h * (BA^{-1}w + z_0) - e^{tA} * \mathcal{K}(Eq, BA^{-1}w + z_0) \\
&\quad + e^{tA} * hBu_0 + e^{tA} * \mathcal{K}(q, CA^{-1}w + z_1) - e^{tA} * \mathcal{M}(Eq, Bu_0) + e^{tA} * \mathcal{M}(q, Cu_0) \\
&=: A^{-1}w_0 + U(w, h, q).
\end{aligned} \tag{4.15}$$

Now we rewrite the fixed point system (3.35), (3.38) in the abstract form

$$\begin{aligned}
h(t) &= h_0(t) - [J_1(u_0)]^{-1} \left\{ \Psi[\mathcal{M}(J_3(u_0)\{\Phi[N_1(v, h, q)(t)] - \Phi_1[v(t)]\}, Cu_0)] \right. \\
&\quad \left. - \Psi[\mathcal{M}(E(J_3(u_0)\{\Phi[N_1(v, h, q)(t)] - \Phi_1[v(t)]\}), Bu_0)] \right. \\
&\quad \left. - \Psi[N_1(v, h, q)(t)] + \Psi_1[v(t)] \right\} \\
&=: h_0(t) + N_3(v, h, q)(t), \quad \forall t \in [0, T],
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
q(t) &= q_0(t) + J_2(u_0)N_3(v, h, q)(t) + J_3(u_0)\{\Phi[N_1(v, h, q)(t)] - \Phi_1[v(t)]\}, \\
&\quad \forall t \in [0, T],
\end{aligned} \tag{4.17}$$

where  $h_0$  and  $q_0$  are the elements appearing in (H11), while (cf. (3.22)) operator  $N_1$  is defined by

$$N_1(v, h, q)(t) = -h * (Bv + z_0)(t) + \mathcal{K}(Eq, Bv + z_0)(t) - \mathcal{K}(q, Cv + z_1)(t). \tag{4.18}$$

**Remark 4.2.** Since  $N_1(v, h, q)(0) = 0$ , from (4.16) and (4.17) we can easily compute the initial values  $\tilde{h}_0$  and  $\tilde{q}_0$  (appearing in (H13)) of functions  $h$  and  $q$ :

$$\begin{cases} \tilde{h}_0 = h_0(0) + J_4(u_0, v_0) = h(0), \\ \tilde{q}_0 = q_0(0) + J_2(u_0)J_4(u_0, v_0) - J_3(u_0)\Phi_1[v_0] = q(0), \end{cases} \tag{4.19}$$

where  $J_4(u_0, v_0)$  is defined by:

$$J_4(u_0, v_0) = [J_1(u_0)]^{-1} \left\{ \Psi[\mathcal{M}(J_3(u_0)\Phi_1[v_0], Cu_0) - \mathcal{M}(EJ_3(u_0)\Phi_1[v_0], Bu_0)] - \Psi_1[v_0] \right\}.$$

**Remark 4.3.** In the explicit case we get the equations

$$\tilde{h}_0 = k_0(R_2), \quad \tilde{q}_0(r) = k'_0(r). \tag{4.20}$$

where  $k_0$  is defined in (3.18).

Introducing the operators

$$\begin{aligned}
\tilde{R}_2(v, h, q) &:= -[J_1(u_0)]^{-1} \left\{ \Psi[\mathcal{M}(J_3(u_0)\Phi[N_1(v, h, q)], Cu_0)] \right. \\
&\quad \left. - \Psi[\mathcal{M}(E(J_3(u_0)\Phi[N_1(v, h, q)], Bu_0)] - \Psi[N_1(v, h, q)] \right\},
\end{aligned} \tag{4.21}$$

$$\tilde{R}_3(v, h, q) := J_2(u_0)\tilde{R}_2(v, h, q) + J_3(u_0)\Phi[N_1(v, h, q)], \quad (4.22)$$

$$\tilde{S}_2(v) := [J_1(u_0)]^{-1} \left\{ \Psi[\mathcal{M}(J_3(u_0)\Phi_1[v], Cu_0)] - \Psi[\mathcal{M}(E(J_3(u_0)\Phi_1[v], Bu_0))] - \Psi_1[v] \right\}, \quad (4.23)$$

$$\tilde{S}_3(v) := J_2(u_0)\tilde{S}_2(v) - J_3(u_0)\Phi_1[v], \quad (4.24)$$

the fixed-point system for  $h$  and  $q$  becomes

$$h = h_0 + \tilde{R}_2(v, h, q) + \tilde{S}_2(v), \quad (4.25)$$

$$q = q_0 + \tilde{R}_3(v, h, q) + \tilde{S}_3(v). \quad (4.26)$$

Therefore, denoting

$$R_2(w, h, q) := \tilde{R}_2(A^{-1}w, h, q), \quad (4.27)$$

$$R_3(w, h, q) := \tilde{R}_3(A^{-1}w, h, q), \quad (4.28)$$

and keeping in mind definitions (4.12) – (4.14), (4.21) – (4.24), thanks to (4.11), (4.25), (4.26) we can pose the following problem related to a given triplet  $(w_0, h_0, q_0) \in C^\beta([0, T]; X) \times C^\beta([0, T]; \mathbb{R}) \times C^\beta([0, T]; Y)$ : *determine a solution  $(w, h, q) \in C^\beta([0, T]; X) \times C^\beta([0, T]; \mathbb{R}) \times C^\beta([0, T]; Y)$  to the fixed-point system*

$$\begin{cases} w = w_0 + R_1(w, h, q) + S_1(q), \\ h = h_0 + R_2(w, h, q) + \tilde{S}_2(A^{-1}w), \\ q = q_0 + R_3(w, h, q) + \tilde{S}_3(A^{-1}w). \end{cases} \quad (4.29)$$

By virtue of (4.8), (4.15) and the linearity of  $\tilde{S}_2, \tilde{S}_3$  it is immediate to check that system (4.29) is equivalent to the following one:

$$\begin{cases} w = w_0 + R_1(w, h, q) + S_1(q), \\ h = [h_0 + \tilde{S}_2(A^{-1}w_0)] + [R_2(w, h, q) + \tilde{S}_2(U(w, hq))] \\ \quad =: \bar{h}_0 + R_5(w, h, q), \\ q = [q_0 + \tilde{S}_3(A^{-1}w_0)] + [R_3(w, h, q) + \tilde{S}_3(U(w, hq))] \\ \quad =: \bar{q}_0 + R_6(w, h, q). \end{cases} \quad (4.30)$$

Hence, replacing  $q$  in  $S_1(q)$  with  $\bar{q}_0 + R_6(w, h, q)$ , and taking advantage of the linearity of operator  $S_1$ , we deduce that the fixed-point system (4.30) is equivalent to the next one:

$$\begin{cases} w = w_0 + S_1(\bar{q}_0) + R_1(w, h, q) + S_1(R_6(w, h, q)) \\ \quad =: \bar{w}_0 + R_4(w, h, q), \\ h = \bar{h}_0 + R_5(w, h, q), \\ q = \bar{q}_0 + R_6(w, h, q). \end{cases} \quad (4.31)$$

Finally, from (4.14) and the definitions of  $\bar{h}_0, \bar{q}_0$  in (4.30) and of  $\bar{w}_0$  in (4.31) we derive the following representation of  $\bar{w}_0, \bar{h}_0, \bar{q}_0$  in terms of  $v_0, h_0, q_0, z_2$ :

$$\begin{cases} \bar{w}_0 = Ae^{tA}v_0 + A(e^{tA} * z_2) + S_1(\bar{q}_0), \\ \bar{h}_0 = h_0 + \tilde{S}_2(e^{tA}v_0 + e^{tA} * z_2), \\ \bar{q}_0 = q_0 + \tilde{S}_3(e^{tA}v_0 + e^{tA} * z_2). \end{cases} \quad (4.32)$$

Since the present situation is analogous to the one in [CL], we can follow the same procedure used there (cf. sections 5 and 6) to get the following local in time existence and uniqueness theorem.

**Theorem 4.4.** *Under assumptions (H1)–(H13) there exists  $T^* \in (0, T)$  such that for any  $\tau \in (0, T^*]$  the fixed-point system (4.31) has a unique solution  $(w, h, q) \in C^\beta([0, \tau]; X) \times C^\beta([0, \tau]; \mathbb{R}) \times C^\beta([0, \tau]; Y)$ .*

An immediate consequence of Theorem 4.4 and of the equivalence result proved in this section is the following corollary.

**Corollary 4.5.** *Under assumptions (H1)–(H13) there exists  $T^* \in (0, T)$  such that for any  $\tau \in (0, T^*]$  problem (4.5), (4.6), (4.25), (4.26) admits a unique solution  $(v, h, q) \in [C^{1+\beta}([0, \tau]; X) \cap C^\beta([0, \tau]; X_2)] \times C^\beta([0, \tau]; \mathbb{R}) \times C^\beta([0, \tau]; Y)$ .*

## 5 Solving the identification problem (3.2)–(3.6), (3.35), (3.38) and proving Theorem 2.7

The basic result of this section is the following theorem.

**Theorem 5.1.** *Let assumptions (1.3), (2.1)–(2.2), (2.26)–(2.31) and condition (2.3) be fulfilled. Moreover assume that the data enjoy the properties (2.33)–(2.39), inequalities (2.46), (2.47) and consistency conditions (C2,H,K), (2.57), (2.58).*

*Then there exists  $T^* \in (0, T]$  such that the identification problem (3.2)–(3.6), (3.35), (3.38) admits a unique solution  $(v, h, q) \in \mathcal{U}_{H,K}^{1,p}(T^*) \times C^\beta([0, T^*]; \mathbb{R}) \times C^\beta([0, T^*]; L^p(R_1, R_2))$  depending continuously on the data with respect to the norms related to the Banach spaces in (2.33)–(2.39).*

*In the case of the specific operators  $\Phi, \Psi$  defined by (1.12), (1.13) the previous result is still true if we assume that  $\lambda \in C^1(\partial B(0, R_2))$  and  $\psi \in C^1(\bar{\Omega})$  with  $\psi = 0$  on the part of  $\partial\Omega$  where the Dirichlet condition is possibly prescribed.*

*Proof.* For any  $p \in (3, +\infty)$  let us choose the Banach spaces  $X, \tilde{X}_1, X_1, X_2, Y, Y_1$  according to the rule

$$X = L^p(\Omega), \quad \tilde{X}_1 = W^{1,p}(\Omega), \quad X_1 = W_{H,K}^{1,p}(\Omega), \quad X_2 = W_{H,K}^{2,p}(\Omega), \quad (5.1)$$

$$Y = L^p(R_1, R_2), \quad Y_1 = W^{1,p}(R_1, R_2), \quad (5.2)$$

where the spaces  $W_{H,K}^{1,p}(\Omega)$  are defined, respectively, in (2.41)–(2.44) with  $\gamma = 1/2$ .

Of course, with this choice the operators  $\mathcal{B}, \mathcal{C}$  defined by (1.2) with  $\mathcal{D}(B) = X_2, \mathcal{D}(C) =$

$X_1$ ,  $Bu = \mathcal{B}u$ ,  $Cu = \mathcal{C}u$ , satisfy assumptions (H4) – (H6).

Let us define  $A$  to be the second-order differential operator  $\mathcal{A} - \lambda_0 I : \mathcal{D}(A) \subset L^p(\Omega) \rightarrow L^p(\Omega)$ ,  $\mathcal{A}$  defined in (1.2) and satisfying (1.3), (2.1) and  $\lambda_0$  being any (fixed) positive constant.

To show that assumptions (H1) – (H3) hold we recall that  $p \in (3, +\infty)$  and reason as in the proof of theorem 7.3.6 in [PA]. For this purpose we assume that  $u \in W_{H,K}^{2,p}(\Omega)$  is a solution to the equation

$$\lambda u - Au = f, \quad f \in L^p(\Omega). \quad (5.3)$$

From the identity

$$\begin{aligned} (\lambda + \lambda_0) \|u\|_{L^p(\Omega)}^p + \int_{\Omega} \sum_{j,k=1}^3 a_{j,k}(x) [(p-1)\gamma_j \gamma_k + \delta_j \delta_k + i(p-2)\gamma_k \delta_j] dx \\ = \int_{\Omega} f(x) |u(x)|^{p-2} \bar{u}(x) dx, \quad f = \lambda u - Au, \quad \operatorname{Re} \lambda \geq 0, \end{aligned} \quad (5.4)$$

where  $|u|^{(p-4)/2} \bar{u} D_k u = \gamma_k + i\delta_k$ ,  $k = 1, 2, 3$ , we easily derive the estimates

$$\operatorname{Re}(\lambda + \lambda_0) \|u\|_{L^p(\Omega)}^p \leq \|f\|_{L^p(\Omega)} \|u\|_{L^p(\Omega)}^{p-1}, \quad \operatorname{Re} \lambda \geq 0, \quad (5.5)$$

$$\int_{\Omega} \sum_{k=1}^3 (|\gamma_k|^2 + |\delta_k|^2) dx \leq \frac{1}{\alpha_1} \|f\|_{L^p(\Omega)} \|u\|_{L^p(\Omega)}^{p-1}, \quad (5.6)$$

$$\begin{aligned} |\operatorname{Im}(\lambda + \lambda_0)| \|u\|_{L^p(\Omega)}^p &\leq (p-2) \frac{\alpha_2}{2} \int_{\Omega} \sum_{k=1}^3 (|\gamma_k|^2 + |\delta_k|^2) dx + \|f\|_{L^p(\Omega)} \|u\|_{L^p(\Omega)}^{p-1} \\ &\leq [(p-2) \frac{\alpha_2}{2} + 1] \|f\|_{L^p(\Omega)} \|u\|_{L^p(\Omega)}^{p-1}, \quad \operatorname{Re} \lambda \geq 0. \end{aligned} \quad (5.7)$$

From (5.5) and (5.7) we deduce

$$|\lambda + \lambda_0| \|u\|_{L^p(\Omega)} \leq \left\{ 1 + [(p-2) \frac{\alpha_2}{2\alpha_1} + 1]^2 \right\}^{1/2} \|f\|_{L^p(\Omega)}, \quad \operatorname{Re} \lambda \geq 0, \quad (5.8)$$

Therefore  $(\lambda I - A)$  is injective and has a closed range in  $L^p(\Omega)$  if  $\operatorname{Re} \lambda \geq 0$ .

To show that  $(\lambda I - A)$  is also surjective if  $\operatorname{Re} \lambda \geq 0$ , let  $v \in L^{p'}(\Omega)$ ,  $p' = p/(p-1)$ , be a function satisfying  $\int_{\Omega} [\lambda u(x) - Au(x)] v(x) dx = 0$  for any  $u \in W_{H,K}^{2,p}(\Omega)$ . From lemma 7.3.4 in [PA], which applies also to our more general case, we deduce that  $A$  is self-adjoint. Hence we get that  $v \in W_{H,K}^{2,p'}(\Omega)$  and  $\int_{\Omega} u(x) [\bar{\lambda} v(x) - Av(x)] dx = 0$  for any  $u \in W_{H,K}^{2,p}(\Omega)$ . Since  $W_{H,K}^{2,p}(\Omega)$  is dense in  $L^p(\Omega)$  we deduce that  $\bar{\lambda} v - Av = 0$  in  $L^{p'}(\Omega)$ ,  $v \in W_{H,K}^{2,p'}(\Omega)$ . Then from the definition  $A = \mathcal{A} - \lambda_0 I$  and the following inequality (cf. formula (7) in [OK])

$$\operatorname{Re} \langle \mathcal{A}v, |v|^{p'-2} \bar{v} \rangle \leq 0, \quad \forall v \in W_{H,K}^{2,p'}(\Omega),$$

where  $p' \in (1, 2)$ , we easily conclude that  $v = 0$ , i.e. the range of  $(\lambda I - A)$  is the entire space  $L^p(\Omega)$ . Therefore  $(\lambda I - A)$  is bijective for all  $\lambda \in \mathbb{C}$  such that  $\operatorname{Re} \lambda \geq 0$  and as a

consequence of (5.8) we have  $\rho(A) \supset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq 0\}$ .

Finally, from proposition 2.1.11 in [LU] and (5.8) we deduce that  $A$  is sectorial and its resolvent satisfies the estimate

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(L^p(\Omega))} \leq \frac{C_1}{|\lambda|}, \quad \forall \lambda \in \Sigma_\zeta, \quad (5.9)$$

for some  $\zeta \in (\pi/2, \pi)$ . Hence (H1) and (H2) hold.

Moreover, from (5.8) with  $\lambda = 0$  and theorem 3.1.1 in [LU] we deduce the estimate

$$\|u\|_{W^{2,p}(\Omega)} \leq C_2 \|Au\|_{L^p(\Omega)}, \quad \forall u \in W_{H,K}^{2,p}(\Omega). \quad (5.10)$$

Let now  $u \in W_{H,K}^{2,p}(\Omega)$  be a solution to the equation (5.3). Then, for any  $\lambda \in \Sigma_\zeta$ , we get

$$\|u\|_{W^{2,p}(\Omega)} \leq C_2 \|Au\|_{L^p(\Omega)} \leq C_2 (\|\lambda\| \|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}) \leq C_2 (C_1 + 1) \|f\|_{L^p(\Omega)}. \quad (5.11)$$

Finally, from the interpolation inequality

$$\|u\|_{W_{H,K}^{1,p}(\Omega)} \leq C_3 \|u\|_{W^{2,p,H,K}(\Omega)}^{1/2} \|u\|_{L^p(\Omega)}^{1/2} \leq \frac{C_3 [C_2 (C_1 + 1)]^{1/2}}{|\lambda|^{1/2}} \|f\|_{L^p(\Omega)}, \quad \forall u \in W_{H,K}^{2,p}(\Omega), \quad (5.12)$$

we obtain that the resolvent  $(\lambda I - A)^{-1}$  belongs to  $\mathcal{L}(X; X_1)$  for any  $\lambda \in \Sigma_\zeta$  and satisfies the estimate

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X; X_1)} \leq C_4 |\lambda|^{-1/2}, \quad \forall \lambda \in \Sigma_\zeta. \quad (5.13)$$

Therefore (H3) is satisfied, too.

Define now the operators  $\Phi, \Phi_1, \Psi, \Psi_1$  respectively by (1.12), (2.68), (1.13), (2.70) and operators  $E$  and  $\mathcal{M}$  by

$$Eq(r) = \int_r^{R_2} q(\xi) d\xi, \quad \forall r \in [R_1, R_2], \quad (5.14)$$

$$\mathcal{M}(q, w)(x) = q(|x|)w(x), \quad \forall x \in \Omega. \quad (5.15)$$

Observe then that by virtue of Hölder's inequality we get

$$\begin{aligned} \int_{R_1}^{R_2} |Eq(r)|^p dr &\leq \int_{R_1}^{R_2} \left[ \int_{R_1}^{R_2} |q(\xi)| d\xi \right]^p dr \leq \|q\|_{L^p(R_1, R_2)}^p \int_{R_1}^{R_2} (R_2 - R_1)^{p-1} dr \\ &= (R_2 - R_1)^p \|q\|_{L^p(R_1, R_2)}^p, \quad \forall q \in L^p(R_1, R_2). \end{aligned} \quad (5.16)$$

Since  $D_r Eq(r) = -q(r)$ , it follows that  $E \in \mathcal{L}(L^p(R_1, R_2), W^{1,p}(R_1, R_2))$ . It is an easy task to show that  $\mathcal{M}$  is a continuous bilinear operator from  $L^p(R_1, R_2) \times W^{1,p}(\Omega)$  to  $L^p(\Omega)$  and from  $W^{1,p}(R_1, R_2) \times L^p(\Omega)$  to  $L^p(\Omega)$ , since the Sobolev imbedding theorems with  $p \in (3, +\infty)$  imply  $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$  and  $W^{1,p}(R_1, R_2) \hookrightarrow C([R_1, R_2])$ . Hence assumptions (H7) and (H8) are satisfied.

Moreover, the assumptions in (H10), but  $J_1(u_0) \neq 0$ , are satisfied according to (2.33) – (2.36). Then, if we define  $J_1(u_0)$ ,  $J_2(u_0)$ ,  $J_3(u_0)$  according to formulae (2.47), (3.39), (3.30), it immediately follows that assumption (H9) is satisfied as well as the condition  $J_1(u_0) \neq 0$  in (H10) is.

Finally we estimate the vector  $(v_0, z_0, z_1, z_2, h_0, q_0)$  in terms of the data  $(u_0, u_1, f, g_1, g_2)$ . Definitions (3.10), (3.11), (3.31), (3.34) imply

$$\begin{aligned} N_1^0(u_1, g_1, f), N_3^0(u_0, u_1, g_1, f) &\in C^\beta([0, T]; L^p(R_1, R_2)), \\ N_2^0(u_1, g_2, f), N_0(u_0, u_1, g_1, g_2, f) &\in C^\beta([0, T]). \end{aligned}$$

Therefore from (3.36) and (3.40) we deduce

$$(h_0, q_0) \in C^\beta([0, T]) \times C^\beta([0, T]; L^p(R_1, R_2)), \quad (5.17)$$

whereas from (4.7) and hypotheses (2.33)–(2.35) it follows

$$(z_0, z_1, z_2) \in C^\beta([0, T]; L^p(\Omega)) \times C^\beta([0, T]; W^{1,p}(\Omega)) \times C^\beta([0, T]; L^p(\Omega)). \quad (5.18)$$

Hence assumptions (H11)–(H12) are also satisfied.

To check condition (H13) first we recall that in this case the interpolation space  $\mathcal{D}_A(\beta, +\infty)$  coincides with the Besov spaces  $B_{H,K}^{2\beta,p,\infty}(\Omega) \equiv (L^p(\Omega), W_{H,K}^{2,p}(\Omega))_{\beta,\infty}$  (cf. [TR, section 4.3.3]). Moreover, we recall that  $B_{H,K}^{2\beta,p,p}(\Omega) = W_{H,K}^{2\beta,p}(\Omega)$ . Finally, we remind the basic inclusion (cf. [TR, section 4.6.1])

$$W^{s,p}(\Omega) \hookrightarrow B^{s,p,\infty}(\Omega), \quad \text{if } s \notin \mathbb{N}. \quad (5.19)$$

Since our function  $F$  defined in (2.37) belongs to  $W_{H,K}^{2\beta,p}(\Omega)$ , it is necessarily an element of  $B_{H,K}^{2\beta,p,\infty}(\Omega)$ . Therefore (H13) is satisfied, too. The proof is now complete.  $\square$

**Proof of Theorem 2.7.** It easily follows from Theorems 3.2 and 5.1.  $\square$

**Remark 5.2.** We want here to give some insight into the somewhat involved condition (2.37). For this purpose we need to assume that the coefficients  $a_{i,j} \in C^{2+\alpha}(\overline{\Omega})$ ,  $i, j = 1, 2, 3$ , for some  $\alpha \in (2\beta, 1)$ ,  $\beta \in (0, 1/2)$ . First, we give a sketch of how the membership of function  $F$  in  $W_{H,K}^{2\beta,p}(\Omega)$  may be derived from (3.18) and the following stricter conditions on the linear operator  $\Phi_1$  appearing on (2.30) and on the data

$$\Phi_1 \in \mathcal{L}(W^{3,p}(\Omega); W^{2,p}(\Omega)), \quad (5.20)$$

$$\mathcal{A}u_0(\cdot) + f(0, \cdot) - D_t u_1(0, \cdot) \in W_{H,K}^{2,p}(\Omega) \cap W^{3,p}(\Omega). \quad (5.21)$$

We observe that in our application the function appearing in (H13) coincide, by virtue of formulae (3.20), (3.21), with function  $F$  defined in (2.37).

To show that  $F$  belongs to  $W_{H,K}^{2\beta,p}(\Omega)$  we limit ourselves to pointing out the following basic steps ensuring that function  $k_0$  defined in (3.18) actually belongs to  $C^{1+\alpha}([R_1, R_2])$ :

- i) for any  $\rho \in C^\alpha(\overline{\Omega})$ ,  $\alpha \in (2\beta, 1)$ ,  $w \in W^{2\beta,p}(\Omega)$ ,  $\rho w \in W^{2\beta,p}(\Omega)$  and satisfies the estimate  $\|\rho w\|_{W^{2\beta,p}(\Omega)} \leq C\|\rho\|_{C^\alpha(\overline{\Omega})}\|w\|_{W^{2\beta,p}(\Omega)}$ ;
- ii) operator  $\Phi$  maps  $C^\alpha(\overline{\Omega})$  into  $C^\alpha([R_1, R_2])$ .

As for as the boundary conditions involved by assumption (H13) are concerned, we observe that they are missing when  $(H,K) = (N,N)$ , while in the remaining case they are so complicated that we like better not to explicit them and we limit to list them as

$F$  satisfies boundary conditions  $(H,K)$ .

Of course, when needed, such conditions can be explicitly computed in terms of the data and function  $k_0$  defined in (3.18).

## 6 The two-dimensional case

In this section we deal with the planar identification problem  $P(H,K)$  related to the annulus  $\Omega = \{x \in \mathbb{R}^2 : R_1 < |x| < R_2\}$ ,  $0 < R_1 < R_2$ .

Operators  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$  are defined by (1.2) simply replacing the subscript 3 with 2:

$$\mathcal{A} = \sum_{j=1}^2 D_{x_j} \left( \sum_{k=1}^2 a_{j,k}(x) D_{x_k} \right), \quad \mathcal{B} = \sum_{j=1}^2 D_{x_j} \left( \sum_{k=1}^2 b_{j,k}(x) D_{x_k} \right), \quad \mathcal{C} = \sum_{j=1}^2 c_j(x) D_{x_j}. \quad (6.1)$$

Moreover, we assume that there exist two positive constants  $\alpha_1$  and  $\alpha_2$  with  $\alpha_1 \leq \alpha_2$  such that

$$\alpha_1 |\xi|^2 \leq \sum_{i,j=1}^2 a_{ij}(x) \xi_i \xi_j \leq \alpha_2 |\xi|^2, \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^2. \quad (6.2)$$

Furthermore we assume that the coefficients of operators  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$  satisfy also the following properties corresponding to (1.20), (2.1), (2.2):

$$a_{i,j} \in W^{2,\infty}(\Omega), \quad a_{i,j} = a_{j,i}, \quad b_{i,j} \in W^{1,\infty}(\Omega), \quad c_i \in L^\infty(\Omega), \quad i, j = 1, 2, \quad (6.3)$$

$$\sum_{j,k=1}^2 x_j x_k a_{j,k}(x) = |x|^2 h(|x|), \quad \forall x \in \overline{\Omega}, \quad (6.4)$$

for some  $h \in C(\overline{\Omega})$ .

In the present case an example of admissible linear operators  $\Phi$  and  $\Psi$  is now the following:

$$\Phi[v](r) := \int_0^{2\pi} \lambda(R_2 x') v(r x') d\varphi, \quad (6.5)$$

$$\Psi[v] := \int_{R_1}^{R_2} r dr \int_0^{2\pi} \psi(r x') v(r x') d\varphi, \quad (6.6)$$

where  $(x_1, x_2) = (r \cos \varphi, r \sin \varphi)$ ,  $x' = (\cos \varphi, \sin \varphi)$ .

From (2.14) we obtain

$$\begin{cases} D_{x_1} = \cos \varphi D_r - \frac{\sin \varphi}{r} D_\varphi, \\ D_{x_2} = \sin \varphi D_r + \frac{\cos \varphi}{r} D_\varphi. \end{cases} \quad (6.7)$$

Therefore, setting  $a_{i,j}(r, \varphi) = a_{i,j}(r \cos \varphi, r \sin \varphi)$ , from (6.7) we deduce

$$\sum_{k=1}^2 a_{1,k}(x) D_{x_k} = f_1(r, \varphi) D_r + \frac{f_2(r, \varphi)}{r} D_\varphi, \quad (6.8)$$

$$\sum_{k=1}^2 a_{2,k}(x) D_{x_k} = g_1(r, \varphi) D_r + \frac{g_2(r, \varphi)}{r} D_\varphi, \quad (6.9)$$

functions  $f_j$ ,  $g_j$ ,  $j = 1, 2$ , being defined by

$$\begin{cases} f_1(r, \varphi) := \tilde{a}_{1,1}(r, \varphi) \cos \varphi + \tilde{a}_{1,2}(r, \varphi) \sin \varphi, \\ f_2(r, \varphi) := \tilde{a}_{1,2}(r, \varphi) \cos \varphi - \tilde{a}_{1,1}(r, \varphi) \sin \varphi, \end{cases} \quad (6.10)$$

$$\begin{cases} g_1(r, \varphi) := \tilde{a}_{2,1}(r, \varphi)\cos\varphi + \tilde{a}_{2,2}(r, \varphi)\sin\varphi, \\ g_2(r, \varphi) := \tilde{a}_{2,2}(r, \varphi)\cos\varphi - \tilde{a}_{2,1}(r, \varphi)\sin\varphi. \end{cases} \quad (6.11)$$

Hence, from (6.7) – (6.9) we get

$$\begin{aligned} D_{x_1} \left( \sum_{k=1}^2 a_{1,k}(x) D_{x_k} \right) &= D_r \left[ f_1(r, \varphi) \cos\varphi D_r + \frac{f_2(r, \varphi) \cos\varphi}{r} D_\varphi \right] \\ &\quad - \frac{\sin\varphi}{r} D_\varphi \left[ f_1(r, \varphi) D_r + \frac{f_2(r, \varphi)}{r} D_\varphi \right], \end{aligned} \quad (6.12)$$

$$\begin{aligned} D_{x_2} \left( \sum_{k=1}^2 a_{2,k}(x) D_{x_k} \right) &= D_r \left[ g_1(r, \varphi) \sin\varphi D_r + \frac{g_2(r, \varphi) \sin\varphi}{r} D_\varphi \right] \\ &\quad - \frac{\cos\varphi}{r} D_\varphi \left[ g_1(r, \varphi) D_r + \frac{g_2(r, \varphi)}{r} D_\varphi \right]. \end{aligned} \quad (6.13)$$

Defining the following functions

$$k_j(r, \varphi) := f_j(r, \varphi) \cos\varphi + g_j(r, \varphi) \sin\varphi, \quad j = 1, 2, \quad (6.14)$$

and using (6.10)–(6.11) we can easily check that, by virtue of (6.4), we have

$$k_1(r, \varphi) = \tilde{a}_{1,1}(r, \varphi) \cos^2\varphi + 2\tilde{a}_{1,2}(r, \varphi) \cos\varphi \sin\varphi + \tilde{a}_{2,2}(r, \varphi) \sin^2\varphi =: h(r). \quad (6.15)$$

Then, rearranging the terms on the right-hand sides of (6.12), (6.13) we obtain the following polar representation for the second order differential operator  $\mathcal{A}$ :

$$\begin{aligned} \tilde{\mathcal{A}} &= D_r \left[ k_1(r) D_r + \frac{k_2(r, \varphi)}{r} D_\varphi \right] - \frac{\sin\varphi}{r} D_\varphi \left[ f_1(r, \varphi) D_r + \frac{f_2(r, \varphi, \theta)}{r} D_\varphi \right] \\ &\quad + \frac{\cos\varphi}{r} D_\varphi \left[ g_1(r, \varphi) D_r + \frac{g_2(r, \varphi)}{r} D_\varphi \right]. \end{aligned} \quad (6.16)$$

**Remark 6.1.** Similarly to the three-dimensional case a class of coefficients  $a_{i,j}$  satisfying property (6.4) is

$$\begin{cases} a_{1,1}(x) = a(|x|) + \frac{x_2^2[c(x) - b(|x|)]}{|x|^2} + \frac{x_1^2 d(|x|)}{|x|^2}, \\ a_{2,2}(x) = a(|x|) + \frac{x_1^2[c(x) - b(|x|)]}{|x|^2} + \frac{x_2^2 d(|x|)}{|x|^2}, \\ a_{1,2}(x) = a_{2,1}(x) = \frac{x_1 x_2 [b(|x|) - c(x) + d(|x|)]}{|x|^2}, \end{cases} \quad (6.17)$$

where  $a, b, d \in C^{2,\infty}([R_1, R_2])$  and  $c \in W^{2,\infty}(\Omega)$ ,  $a$  and  $c$  being, respectively, *positive* and *non-negative* functions such that

$$a(r) - b^+(r) - d^-(r) > 0, \quad \forall r \in [R_1, R_2]. \quad (6.18)$$

This property ensure the uniform ellipticity of  $\mathcal{A}$ .

Working in Sobolev spaces related to  $L^p(\Omega)$  with

$$p \in (2, +\infty) \quad (6.19)$$

we note that our requirements on operators  $\Phi$  and  $\Psi$  and the data are the same as in (2.27)–(2.39) whereas the Banach spaces  $\mathcal{U}^{s,p}(T)$  and  $\mathcal{U}_{H,K}^{s,p}(T)$  are still defined by (2.52).

**Theorem 6.2.** *Let assumptions (6.2) – (6.4), (6.19), (2.27) – (2.31) be fulfilled. Moreover assume that the data enjoy the properties (2.33) – (2.39) and satisfy inequalities (2.46), (2.47).*

*Then there exists  $T^* \in (0, T]$  such that the identification problem  $P(H, K)$  ( $H, K \in \{D, N\}$ ), admits a unique solution  $(u, k) \in \mathcal{U}_{H,K}^{2,p}(T^*) \times C^\beta([0, T^*], W^{1,p}(R_1, R_2))$  depending continuously on the data with respect to the norms pointed out in (2.33) – (2.39).*

*In the case of the specific operators  $\Phi, \Psi$  defined as in (6.5), (6.6) the previous result is still true if we assume  $\lambda \in C^1(\partial B(0, R_2))$  and  $\psi \in C^1(\overline{\Omega})$  with  $\psi = 0$  on the part of  $\partial\Omega$  where the Dirichlet condition is possibly prescribed.*

**Lemma 6.3.** *When  $\Phi$  and  $\Psi$  are defined by (6.5) and (6.6), respectively, conditions (2.27) – (2.31) are satisfied under assumptions (6.3), (6.4) on the coefficients  $a_{i,j}$  ( $i, j = 1, 2$ ) and the hypotheses that  $\lambda \in C^1(\partial B(0, R_2))$  and  $\psi \in C^1(\overline{\Omega})$  with  $\psi = 0$  on the part of  $\partial\Omega$  where the Dirichlet condition is possibly prescribed.*

*Proof.* It is essentially the same as that of Lemma 2.8. Therefore, we leave it to the reader.  $\square$

## 7 Solving system (1.23) and (1.24)

We solve here the following integro-differential system introduced in remark 1.1, where  $n = 2, 3$ :

$$\begin{aligned} \int_0^t \{ D_r k(t-s, r) D_t D_r u(s, r) + k(t-s, r) [D_t D_r^2 u(s, r) + (n-1)r^{-1} D_t D_r u(s, r)] \} ds \\ + D_r k(t, r) D_r u(0, r) + k(t, r) [D_r^2 u(0, r) + (n-1)r^{-1} D_r u(0, r)] = D_t \tilde{f}(t, r), \\ \forall (t, r) \in [0, T] \times [R_1, R_2], \end{aligned} \quad (7.1)$$

$$\int_{R_1}^{R_2} \lambda(r) k(t, r) dr = g(t), \quad \forall t \in [0, T]. \quad (7.2)$$

We assume that the data  $(u, f, g)$  enjoy the following properties:

$$u \in W^{1,1}((0, T); W^{2,p}(R_1, R_2)), \quad (7.3)$$

$$\tilde{f} \in C^1([0, T]; L^p(R_1, R_2)), \quad (7.4)$$

$$g \in C([0, T]; \mathbb{R}), \quad (7.5)$$

$$\lambda \in L^{p'}(R_1, R_2), \quad (7.6)$$

where  $p \in [1, \infty]$  and  $p'$  denotes the conjugate exponent of  $p$ .

Like in section 3 we introduce the new unknowns

$$h(t) = k(t, R_1), \quad q(t, r) = D_r k(t, r), \quad \forall (t, r) \in [0, T] \times [R_1, R_2] \quad (7.7)$$

and express  $k$  in terms of  $h$  and  $q$ :

$$k(t, r) = h(t) + \int_{R_1}^r q(t, \rho) d\rho, \quad \forall (t, r) \in [0, T] \times [R_1, R_2]. \quad (7.8)$$

Changing the order of integration, from (7.2) and (7.8) we immediately derive the equation

$$h(t) \int_{R_1}^{R_2} \lambda(r) dr + \int_{R_1}^{R_2} \lambda_1(r) q(t, r) dr = g(t), \quad \forall t \in [0, T], \quad (7.9)$$

where

$$\lambda_1(r) = \int_r^{R_2} \lambda(\rho) d\rho, \quad \forall r \in [R_1, R_2]. \quad (7.10)$$

Assuming

$$\kappa^{-1} := \int_{R_1}^{R_2} \lambda(r) dr \neq 0, \quad (7.11)$$

from (7.9) we easily deduce

$$h(t) = \kappa g(t) - \kappa \int_{R_1}^{R_2} \lambda_1(\rho) q(t, \rho) d\rho, \quad \forall t \in [0, T]. \quad (7.12)$$

Assume now

$$|D_r u(0, r)| \geq m > 0, \quad \forall r \in [R_1, R_2]. \quad (7.13)$$

Then, owing to (7.8) and (7.12), system (7.1), (7.2) is equivalent to the following Volterra integral equation of the second kind:

$$\begin{aligned} q(t, r) - \kappa \alpha(r) \int_{R_1}^{R_2} \lambda_1(\rho) q(t, \rho) d\rho + \alpha(r) \int_{R_1}^r q(t, \rho) d\rho + \int_0^t \beta(t-s, r) q(s, r) ds \\ - \kappa \int_0^t \gamma(t-s, r) ds \int_{R_1}^{R_2} \lambda_1(\rho) q(s, \rho) d\rho + \int_0^t \gamma(t-s, r) ds \int_{R_1}^r q(s, \rho) d\rho = \tilde{f}_1(t, r), \\ \forall (t, r) \in [0, T] \times [R_1, R_2], \end{aligned} \quad (7.14)$$

where

$$\alpha(r) = \frac{D_r^2 u(0, r) + (n-1)r^{-1} D_r u(0, r)}{D_r u(0, r)}, \quad (7.15)$$

$$\beta(t, r) = \frac{D_t D_r u(t, r)}{D_r u(0, r)}, \quad (7.16)$$

$$\gamma(t, r) = \frac{D_t D_r^2 u(t, r) + (n-1)r^{-1} D_t D_r u(t, r)}{D_r u(0, r)}, \quad (7.17)$$

$$\tilde{f}_1(t, r) = \frac{D_t \tilde{f}(t, r)}{D_r u(0, r)} - \kappa g(t) \alpha(r) - \kappa \int_0^t \gamma(t-s, r) g(s) ds. \quad (7.18)$$

From assumption (7.3)–(7.5) we easily deduce that

$$\alpha \in L^p(R_1, R_2), \quad \beta \in L^1((0, T); C([R_1, R_2])), \quad \gamma \in L^1((0, T); L^p(R_1, R_2)), \quad (7.19)$$

$$\tilde{f}_1 \in C([0, T]; L^p(R_1, R_2)). \quad (7.20)$$

Moreover  $\alpha, \beta, \gamma, \tilde{f}_1$  satisfy the estimates

$$\|\alpha\|_{L^p(R_1, R_2)} \leq \frac{1}{m} \left(1 + \frac{n-1}{R_1}\right) \|u\|_{W^{1,1}((0, T); W^{2,p}(R_1, R_2))}, \quad (7.21)$$

$$\|\beta(t, \cdot)\|_{L^p(R_1, R_2)} \leq \frac{1}{m} \|D_t u(t, \cdot)\|_{W^{2,p}(R_1, R_2)}, \quad (7.22)$$

$$\|\gamma(t, \cdot)\|_{L^p(R_1, R_2)} \leq \frac{1}{m} \left(1 + \frac{n-1}{R_1}\right) \|D_t u(t, \cdot)\|_{W^{2,p}(R_1, R_2)}, \quad (7.23)$$

$$\begin{aligned} \|\tilde{f}_1(t, \cdot)\|_{L^p(R_1, R_2)} &\leq \frac{1}{m} \|D_t \tilde{f}(t, \cdot)\|_{L^p(R_1, R_2)} + |\kappa| \|g\|_{C([0, T]; \mathbb{R})} \|\alpha\|_{L^p(R_1, R_2)} \\ &\quad + |\kappa| \|g\|_{C([0, T]; \mathbb{R})} \int_0^T \|\gamma(s, \cdot)\|_{L^p(R_1, R_2)} ds. \end{aligned} \quad (7.24)$$

Consider now the auxiliary integral equation

$$q(t, r) + \alpha(r) \int_{R_1}^r q(t, \rho) d\rho = f(t, r) + \kappa \alpha(r) \int_{R_1}^{R_2} \lambda_1(\rho) q(t, \rho) d\rho. \quad (7.25)$$

Hence, setting

$$Q(t, r) = \int_{R_1}^r q(t, \rho) d\rho, \quad (7.26)$$

$$z(t, r) = f(t, r) + \kappa \alpha(r) \int_{R_1}^{R_2} \lambda_1(\rho) q(t, \rho) d\rho, \quad (7.27)$$

it turns out that (7.25) is equivalent to the following Cauchy problem

$$\begin{cases} D_r Q(t, r) + \alpha(r) Q(t, r) = z(t, r), & \forall r \in (R_1, R_2), \\ Q(t, R_1) = 0, \end{cases} \quad (7.28)$$

which has the solution

$$Q(t, r) = \int_{R_1}^r z(t, \rho) \exp\left(\int_r^\rho \alpha(s) ds\right) d\rho. \quad (7.29)$$

Therefore, using (7.26), (7.27) and replacing the expression for  $Q$  in (7.29) into (7.25), we get

$$q(t, r) = f_1(t, r) + \kappa \alpha(r) \left( \int_{R_1}^{R_2} \lambda_1(\rho) q(t, \rho) d\rho \right) \exp\left(\int_r^{R_1} \alpha(s) ds\right), \quad (7.30)$$

where

$$f_1(t, r) = f(t, r) - \alpha(r) \int_{R_1}^r f(t, \rho) \exp\left(\int_r^\rho \alpha(s) ds\right) d\rho. \quad (7.31)$$

Multiplying then each side in (7.30) by  $\lambda_1(r)$  and integrating over  $[R_1, R_2]$ , we easily derive the equation

$$\left[1 - \kappa \int_{R_1}^{R_2} \lambda_1(\rho) \alpha(\rho) \exp\left(\int_{\rho}^{R_1} \alpha(s) ds\right) d\rho\right] \int_{R_1}^{R_2} \lambda_1(r) q(t, r) dr = \int_{R_1}^{R_2} \lambda_1(r) f_1(t, r) dr, \quad \forall t \in [0, T]. \quad (7.32)$$

By an integration by parts, which makes use of condition (7.10) and (7.11) we easily deduce the equality

$$1 - \kappa \int_{R_1}^{R_2} \lambda_1(\rho) \alpha(\rho) \exp\left(\int_{\rho}^{R_1} \alpha(s) ds\right) d\rho = \kappa \int_{R_1}^{R_2} \lambda(\rho) \exp\left(\int_{\rho}^{R_1} \alpha(s) ds\right) d\rho. \quad (7.33)$$

Assume now

$$\kappa_1 = \int_{R_1}^{R_2} \lambda(\rho) \exp\left(\int_{\rho}^{R_1} \alpha(s) ds\right) d\rho \neq 0. \quad (7.34)$$

Then from (7.32) – (7.34) we deduce

$$\int_{R_1}^{R_2} \lambda_1(\rho) q(t, \rho) d\rho = (\kappa \kappa_1)^{-1} \int_{R_1}^{R_2} \lambda_1(\rho) f_1(t, \rho) d\rho, \quad \forall t \in [0, T]. \quad (7.35)$$

Therefore, by easy computations, from (7.30) and (7.31) it follows that the solution to (7.25) is given by

$$q(t, r) = f(t, r) + \int_{R_1}^{R_2} G(r, \rho) f(t, \rho) d\rho, \quad \forall t \in [0, T], \quad (7.36)$$

where the Green function  $G$  is defined as follows

$$G(r, \rho) = \begin{cases} \alpha(r) \left[ \exp\left(\int_r^{R_1} \alpha(s) ds\right) \int_{\rho}^{R_2} \frac{\lambda(\sigma)}{\kappa_1} \exp\left(\int_{\sigma}^{\rho} \alpha(s) ds\right) d\sigma - \exp\left(\int_r^{\rho} \alpha(s) ds\right) \right], & R_1 \leq \rho < r \leq R_2, \\ \alpha(r) \exp\left(\int_r^{R_1} \alpha(s) ds\right) \int_{\rho}^{R_2} \frac{\lambda(\sigma)}{\kappa_1} \exp\left(\int_{\sigma}^{\rho} \alpha(s) ds\right) d\sigma, & R_1 \leq r < \rho \leq R_2. \end{cases} \quad (7.37)$$

Consequently, (7.14) turns out to be equivalent to

$$q(t, r) = \tilde{f}_1(t, r) + \int_{R_1}^{R_2} G(r, \rho) \tilde{f}_1(t, \rho) d\rho + Lq(t, r) + \int_0^t ds \int_{R_1}^{R_2} G_1(t-s, r, \rho) q(s, \rho) d\rho, \quad (7.38)$$

where, denoted with  $\chi_{[R_1, r]}$  the characteristic function of the interval  $[R_1, r]$ , the operator  $L$  and the function  $G_1$  are defined, respectively, by the following formulae

$$Lq(t, r) = - \int_0^t \beta(t-s, r)q(s, r)ds, \quad (7.39)$$

$$\begin{aligned} G_1(t, r, \rho) = & \kappa\gamma(t, r)\lambda_1(\rho) - \gamma(t, r)\chi_{[R_1, r]}(\rho) - \beta(t, \rho)G(r, \rho) \\ & + \kappa\lambda_1(\rho) \int_{R_1}^{R_2} G(r, \xi)\gamma(t, \xi)d\xi - \int_{\rho}^{R_2} G(r, \xi)\gamma(t, \xi)d\xi. \end{aligned} \quad (7.40)$$

Observe now that, according to (7.37),  $G$  satisfies the inequality

$$|G(r, \rho)| \leq C_1|\alpha(r)| \quad \forall r, \rho \in (R_1, R_2) \quad (7.41)$$

where  $C_1 = C_1(p, \kappa_1, \|\alpha\|_{L^1(R_1, R_2)}, \|\lambda\|_{L^1(R_1, R_2)}) > 0$ .

Likewise  $G_1$  satisfies the inequality

$$\begin{aligned} |G_1(t, r, \rho)| \leq & \left[1 + |\kappa|\|\lambda\|_{L^1(R_1, R_2)}\right] \left[|\gamma(t, r)| + C_1|\alpha(r)|\|\gamma(t, \cdot)\|_{L^p(R_1, R_2)}(R_2 - R_1)^{1/p'}\right] \\ & + C_1|\alpha(r)|\|\beta(t, \cdot)\|_{L^\infty(R_1, R_2)}. \end{aligned} \quad (7.42)$$

From (7.42) we easily deduce that

$$G_1 \in L^1((0, T); L^p((R_1, R_2); L^{p'}(R_1, R_2))) \quad (7.43)$$

and satisfy the estimate

$$\begin{aligned} \|G_1(t, \cdot, \cdot)\|_{L^p((R_1, R_2); L^{p'}(R_1, R_2))} & \leq C_1\|\alpha\|_{L^p(R_1, R_2)}\|\beta(t, \cdot)\|_{L^\infty(R_1, R_2)}(R_2 - R_1)^{1/p'} \\ & + \left[1 + |\kappa|\|\lambda\|_{L^1(R_1, R_2)}\right] \left[1 + C_1\|\alpha\|_{L^p(R_1, R_2)}(R_2 - R_1)^{1/p'}\right] \|\gamma(t, \cdot)\|_{L^p(R_1, R_2)}(R_2 - R_1)^{1/p'} \\ & := l(t). \end{aligned} \quad (7.44)$$

According to properties (7.19), (7.20) we easily deduce

$$l \in L^1((0, T); \mathbb{R}). \quad (7.45)$$

Observe also that function

$$w(t, r) = \tilde{f}_1(t, r) + \int_{R_1}^{R_2} G(r, \rho)\tilde{f}_1(t, \rho)d\rho \quad (7.46)$$

belongs to  $C([0, T]; L^p(R_1, R_2))$ . Moreover  $w$  satisfies the inequalities

$$\begin{aligned} |w(t, r)| \leq & |\tilde{f}_1(t, r)| + C_1|\alpha(r)|\|\tilde{f}_1(t, \cdot)\|_{L^p(R_1, R_2)}(R_2 - R_1)^{1/p'}, \\ & \forall (t, r) \in (0, T) \times (R_1, R_2), \end{aligned} \quad (7.47)$$

and

$$\|w(t, \cdot)\|_{L^p(R_1, R_2)} \leq \left[1 + C_1\|\alpha\|_{L^p(R_1, R_2)}(R_2 - R_1)^{1/p'}\right] \|\tilde{f}_1(t, \cdot)\|_{L^p(R_1, R_2)}, \quad \forall t \in (0, T). \quad (7.48)$$

Introduce now the Banach space  $X = L^p(R_1, R_2)$  endowed with the usual norm. Then we can rewrite the integral equation (7.38) as the fixed-point equation

$$q = w + L_1 q \quad (7.49)$$

where, according to (7.38), the operator  $L_1$  is defined via the following formula:

$$L_1 q(t, r) = Lq(t, r) + \int_0^t ds \int_{R_1}^{R_2} G_1(t-s, r, \rho) q(s, \rho) d\rho. \quad (7.50)$$

Observe now that  $L_1$  maps  $C([0, T]; X)$  into itself and satisfies the following inequalities

$$\begin{aligned} \|L_1 q(t)\|_X &\leq \int_0^t \left[ \|\beta(t-s, \cdot)\|_{C([R_1, R_2])} + \|G_1(t-s, \cdot, \cdot)\|_{L^p((R_1, R_2); L^{p'}(R_1, R_2))} \right] \|q(s)\|_X ds \\ &\leq \int_0^t \varphi(t-s) \|q(s)\|_X ds, \quad \forall t \in (0, T), \end{aligned} \quad (7.51)$$

where (cf. (7.44))

$$\varphi(t) = \|\beta(t, \cdot)\|_{C([R_1, R_2])} + l(t), \quad \forall t \in (0, T), \quad (7.52)$$

belongs to  $L^1(0, T)$  according to properties (7.19) and (7.45).

To derive the estimate relative to  $G_1$  observe that, owing to the integral version of Minkowski's inequality, we get

$$\begin{aligned} \left\| \int_0^t ds \int_{R_1}^{R_2} G_1(t-s, \cdot, \rho) q(s, \rho) d\rho \right\|_X &\leq \left\| \int_0^t \|q(s)\|_X \left( \int_{R_1}^{R_2} |G_1(t-s, \cdot, \rho)|^{p'} d\rho \right)^{1/p'} ds \right\|_X \\ &\leq \int_0^T \|q(s)\|_X \|G_1(t-s, \cdot, \cdot)\|_{L^p((R_1, R_2); L^{p'}(R_1, R_2))} ds. \end{aligned} \quad (7.53)$$

We introduce now in  $C([0, T]; X)$  the following weighted norm

$$\|f\|_\sigma = \sup_{t \in [0, T]} e^{-\sigma t} \|f(t)\|_X \quad (7.54)$$

which is equivalent to the usual one.

Now from (7.51) rewritten in the equivalent form

$$e^{-\sigma t} \|L_1 q(t)\|_X \leq \int_0^t e^{-\sigma(t-s)} \varphi(t-s) e^{-\sigma s} \|q(s)\|_X ds, \quad \forall t \in (0, T), \quad (7.55)$$

and Young's theorem on convolution we deduce the basic estimate

$$\|L_1 q\|_\sigma \leq \|q\|_X \int_0^T e^{-\sigma t} \varphi(t) dt. \quad (7.56)$$

Consequently, the norm of the linear operator  $L_1$  does not exceed  $\int_0^T e^{-\sigma t} \varphi(t) dt$ , which tends to 0 as  $\sigma \rightarrow +\infty$ . Therefore, if we choose a large enough  $\sigma$ , then  $I - L_1$  is continuously invertible according to Neumann's theorem.

We have thus proved the following theorem

**Theorem 7.1.** *let  $u, \tilde{f}, g, \lambda$  satisfy properties (7.3) – (7.6) and let assumptions (7.11), (7.13), (7.34) be fulfilled. Then problem (7.1), (7.2) admits a unique solution  $k \in C([0, T]; W^{1,p}(R_1, R_2))$  continuously depending on  $(u, \tilde{f}, g)$  with respect to the norms pointed out.*

## References

- [AD] Adams R. A.: Sobolev Spaces, Academic Press, New York-San Francisco-London 1975.
- [CL] Colombo F., Lorenzi A.: *An identification problem related to parabolic integrodifferential equations with non commuting spatial operators*, J. Inverse Ill Posed Problems, 8 (2000), 505–540.
- [JJ] Janno J.: *An inverse problem arising in compression of visco-elastic medium*, Proc. Estonian Acad. Sci. Phys. Math. 49 (2000), 75-89.
- [JW] Janno J., v. Wolfersdorf L.: *An inverse problem for identification of a time- and space-dependent memory kernel of a special kind in heat conduction*, Inv. Prob. **15** (1999), pp. 1455-1467.
- [LU] A. Lunardi: Analytic semigroups and optimal regularity in parabolic problems, Birkhäuser Verlag, Basel 1995.
- [OK] Okazawa N.: *Sectorialness of second order elliptic operators in divergence form*, Proc. Amer. Math. Soc. 113 (1991), 701-706.
- [PA] A. Pazy: Semigroups of linear operators and applications to partial differential equations, Applied mathematical sciences vol. 44, Springer-Verlag, New York 1983.
- [TR] Triebel, H.: Interpolation Theory, Function Spaces, Differential Operators, North Holland Publ. Co., Amsterdam - New York - Oxford 1978.